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On the Two-Dimensional Dynamical Ising Model In the Phase Coexistence Region

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We consider a Glauber dynamics reversible with respect to the two-dimensional Ising model in a finite square of side L, in the absence of an external field and at large inverse temperature β. We first consider the gap in the spectrum of the generator of the dynamics in two different cases: with plus and open boundary conditions. We prove that, when the symmetry under global spin flip is broken by the boundary conditions, the gap is much larger than the case in which the symmetry is present. For this latter we compute exactly the asymptotics of $-(1/\beta L)\log(\text{gap})$ as $L \to \infty$ and show that it coincides with the surface tension along one of the coordinate axes. As a consequence we are able to study quite precisely the large deviations in time of the magnetization and to obtain an upper bound on the spin–spin time correlation in the infinite-volume plus phase. Our results establish a connection between the dynamical large deviations and those of the equilibrium Gibbs measure studied by Shlosman in the framework of the rigorous description of the Wulff shape for the Ising model. Finally we show that, in the case of open boundary conditions, it is possible to rescale the time with L in such a way that, as $L \to \infty$, the finite-dimensional distributions of the time-rescaled magnetization converge to those of a symmetric continuous-time Markov chain on the two-state space $\{-m^*(\beta), m^*(\beta)\}$, $m^*(\beta)$ being the spontaneous magnetization. Our methods rely upon a novel combination of techniques for bounding from below the gap of symmetric Markov chains on complicated graphs, developed by Jerrum and Sinclair in their Markov chain approach to hard computational problems, and the idea of introducing "block Glauber dynamics" instead of the standard single-site dynamics, in order to put in evidence more effectively the effect of the boundary conditions in the approach to equilibrium.

KEY WORDS: Ising model; phase coexistence region.

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1. INTRODUCTION

In recent years there has been very important progress in the rigorous analysis of Glauber dynamics (see Section 1 for a precise definition) for classical lattice spin systems when the thermodynamic parameters are such that the static system, described by the usual Gibbs measure

$$\mu = \frac{\exp(-\beta H)}{Z}$$

does not undergo a phase transition in the thermodynamic limit.

In particular we refer the reader to the series of papers by Stroock and Zegarlinski (see ref. 28 and references therein), by Olivieri and Martinelli,[18,19] Martinelli *et al.*,[20] and Lu and Yau[16] for the proof, under various mixing conditions on the Gibbs measure μ, of the exponential (in time) relaxation to equilibrium, represented by μ itself, in finite or infinite volume, of the associated Glauber dynamics, and to the work by Schonmann (see ref. 24 and references therein), Kotecky and Olivieri,[9,10] and Scoppola[23] for detailed description of the metastable behavior of Glauber dynamics for Ising-type models close to the line of first-order phase transition.

It is important to emphasize that some of the results in the above works cover most of the one-phase region, going sometimes, e.g., in ferromagnetic systems, arbitrarily close to the critical point.[18,20]

A natural question arises as to what happens when the thermodynamic parameters are such that we do have a phase transition in the thermodynamic limit. To be more concrete, let us consider the usual Ising model in d dimensions, $d \geqslant 2$, in the absence of an external field, described by the formal (normalized) energy function

$$H(\sigma) = -\tfrac{1}{2} \sum_{\substack{x,\, y \in \Lambda^d \\ |x-y|=1}} (\sigma(x)\, \sigma(y) - 1), \qquad \sigma \in \{-1, 1\}^{\mathbf{Z}^d} \tag{0.1}$$

and let us suppose that the inverse temperature β is larger (actually in all rigorous results much larger) than the critical value β_c.

Then, as is well known (see, e.g., ref. 13), any associated infinite-volume Glauber dynamics is not ergodic and it is rather natural to ask how this absence of ergodicity is reflected if we look at the dynamics in a finite, but large cube V_L of side L, where ergodicity is never broken.

A first partial answer was provided by Thomas[29] some years ago. He showed that, if the symmetry of $H(\sigma)$ under global spin flip is not broken by the boundary conditions on the exterior of the cube V_L, then the relaxation time to equilibrium, which in a first approximation can be taken equal to the inverse of the gap in the spectrum of the generator L_{V_L} of the dynamics, diverges, as $L \to \infty$, at least as an exponential of the *surface* L^{d-1}.

The reason for such a result is the following. When the symmetry is not broken, e.g., when boundary conditions are open (i.e., absent) or periodic, then the energy landscape determined by the function $H(\sigma)$ has only two absolute minima, corresponding to the two configurations identically equal to either $+1$ or to -1. Thus the dynamics started, e.g., from all minuses, in order to relax to equilibrium, has to reach the neighborhood of the opposite minimum by necessarily crossing the set of configurations of zero magnetization. Since the Gibbs measure gives to the latter a weight of the order of a negative exponential of the *surface* (see, e.g., ref. 25), a kind of bottleneck is present and the result follows by a rather simple argument (see the first part of the proof of Theorem 4.1 below).

The same reasoning also suggests that, if the symmetry is broken by the boundary conditions, e.g., by fixing equal to $+1$ all spins outside V_L, then the relaxation time should be much shorter than in the previous case since there should be no bottlenecks to cross. Equilibrium is, in this case, induced by the boundary conditions by means of some sort of plus spin wave, initially attached to the boundary and shrinking to zero as time goes on.

The interesting but unproven conjecture is that, at least in two dimensions with plus boundary conditions, the relaxation time will diverge, as $L \to \infty$, like L^2. The proof of the above conjecture would have some very nice consequences on the equal site time correlation function of the infinite-volume dynamics started in the plus phase, for which Fisher and Huse[6] predicted, using the above conjecture, a stretched exponential decay of the form $\exp(-\sqrt{t})$ (see also ref. 21 for numerical simulations and ref. 17 for further discussion).

In this paper we consider the above and other related questions for the two-dimensional Ising model at very low temperature without external field. For some less precise results in arbitrary dimensions see the remark after Theorem 4.2.

In Section 3 we prove a *lower* bound on the gap in the spectrum of the generator L_{V_L} of the Glauber dynamics with plus boundary conditions of the form

$$\text{gap}(L_{V_L}) \geqslant \exp(-C\beta L^{1/2+\varepsilon}), \qquad \varepsilon \in (0, \tfrac{1}{2}] \tag{0.2}$$

which, although it gives a bound on the relaxation time which is far from the conjectured L^2 law, is in any case much larger than the *upper bound* obtained by Thomas without the plus boundary conditions.

As a consequence we derive an upper bound of the form

$$\exp[-\log(t)^{\alpha}], \qquad \alpha \in [0, 2)$$

on the equal site time correlation function of the infinite-volume dynamics started in the plus phase.

In Section 4 we compute exactly the asymptotics of the gap with open boundary conditions. More precisely we obtain, for any $\varepsilon \in (0, \frac{1}{4}]$, any β large enough, and any L

$$\exp[-\beta\tau(\beta)\,L - C\beta L^{1/2+\varepsilon}] \leqslant \text{gap}(L_{V_L}) \leqslant \exp[-\beta\tau(\beta)\,L + C\beta L^{1/2+\varepsilon}]$$

where $\tau(\beta)$ is the surface tension in the direction of, e.g., the horizontal axis. As a byproduct of the proof of this result, we show that the bound (0.2) is valid even if the plus boundary conditions are added on only one side of the square V_L.

The proofs of the above two results follow two very similar steps:

Step 1: We prove the sought result for a generalized Glauber dynamics in which single sites are replaced by suitable blocks. This mean that, given *a priori* a covering $\{Q_i\}$ of V_L, at each updating of the dynamics the spin configuration is changed in only one block Q_i and there it is replaced by the equilibrium Gibbs measure of the block given the configuration outside it. It turns out that a convenient choice of the blocks in our case consists of long and thin overlapping rectangles with basis L and height $L^{1/2+\varepsilon}$, $0 < \varepsilon \ll 1$.

Step 2: We relate the gap of the single-site Glauber dynamics to that of the generalized block dynamics in such a way that the estimates obtained in step 1 are not significantly changed.

The above way to attack the problem is not entirely new; it was in fact introduced long ago by Holley[8] to prove exponential convergence to equilibrium in the one-phase region. One has in fact that, if the system is away from the phase transition region and if the blocks are overlapping large enough (depending on the thermodynamic parameters) cubes, the block Glauber dynamics behaves as a very high-temperature single-site Glauber dynamics, i.e., an almost independent system for which the discussion of the approach to equilibrium is a relatively easy task (see Section 4 of ref. 20).

While we accomplish the first step via a very natural probabilistic construction, the second, rather crucial, step is carried out via the application to our context of a clever geometric technique introduced by Jerrum and Sinclair[11,12,27] (see also ref. 5), to estimate from *below* the gap in the spectrum of a symmetric Markov chain on complicated graphs. They invented their technique while working on a stochastic algorithm approach to compute the partition function Z of the Ising model and the permanent of a large matrix in a time *polynomial* in the size of the problem.

Such a technique, which is illustrated in our case in a self-contained way in Section 2, gives in a very natural way a *lower* bound on the gap of the generator of the dynamics in a rectangle R with shortest side l, with or without boundary conditions, of the form

$$\text{gap}(L_R) \geqslant \exp(-\beta Cl)$$

Morevoer, if the blocks Q_i of the generalized Glauber dynamics are suitable translations of the rectangle R, then

$$\text{gap(Glauber)} \geqslant \exp(-\beta Cl)\ \text{gap(generalized Glauber)}$$

It is worthwhile to mention that our proof of step 1 is constructive in the sense that it indicates how the system actually reaches equilibrium: by simply propagating the plus boundary conditions in the bulk if these are present and the initial configuration is, e.g., all minuses, or by creating inside the starting phase, via a large fluctuation, an almost horizontal (or vertical) interface close, e.g., to the bottom side of V_L which afterward rigidly moves to the opposite side until the other phase has invaded the whole volume.

Once we have a precise control on the relaxation time with open boundary conditions, we can study in detail the large fluctuations of the magnetization

$$m(\sigma_t) = \frac{1}{L^2} \sum_x \sigma_t(x)$$

by considering, for example, the hitting time τ_ρ of the set

$$M_\rho = \{\sigma; m(\sigma) = \rho\}, \qquad \rho \in \{-m^*(\beta), m^*(\beta)\}$$

with $m^*(\beta)$ the spontaneous magnetization.

In Section 5 we show that, if the starting configuration is distributed according to the equilibrium measure restricted to the "phase" of positive magnetization or if it is identically equal to plus one, then the expected value $E(\tau_\rho)$ of the hitting time τ_ρ is of the order

$$E(\tau_\rho) \approx \exp[\beta L \psi(\rho \vee 0)]$$

where the rate function $\psi(\rho)$ is the same as for the static problem:

$$\mu_{V_L}(m(\sigma) = \rho) \approx \exp[-\beta L \psi(\rho)]$$

and it has been computed by Shlosman[26] in the framework of the rigorous description of the Wulff shape for the Ising model carried out by Dobrushin *et al.*[4] We also show that the hitting time τ_ρ rescaled by roughly its average converges, as $L \to \infty$, to an exponential time of mean one.

It is important to outline that the typical configurations of the equilibrium Gibbs measure under the condition $\{m(\sigma) = \rho\}$ have a very precise geometric structure related to the Wulff shape with open boundary conditions.[26] Thus, the fact that the rate function for $E(\tau_\rho)$ is the same as in the static problem suggests that when the system started in the positively magnetized "phase" reaches for the first time the set M_ρ, it does it by forming a droplet of the right volume and with the correct Wulff shape. We hope to come back in a future work to this and related problems.

A key step in the discussion of the above problems is the proof, based on the results of Section 3, that the relaxation time inside a single "phase" is much shorter than the typical values of the hitting time τ_ρ (see Proposition 5.2 for a precise statement).

This last result indicates that the gap in the spectrum of the generator restricted to the invariant subspace of the functions even with respect to global spin flip is much larger than the true gap; unfortunately we do not have any precise statement in this direction.

Finally in Section 6 we complete the analysis of the time evolution of the magnetization by showing that, if the time is scaled with L in such a way that on the new unit of time the system is likely to have jumped from one phase to the other, then the finite-dimensional distributions of the time-scaled magnetization converge, as $L \to \infty$, to those of a continuous-time Markov chain on the two-state space $\{-m^*(\beta), m^*(\beta)\}$ with unitary jump rate.

The rest of the paper contains a preliminary section, Section 1, where all the necessary definitions are given together with the required results on Wulff shape, cluster expansion, and so forth. The proofs of various technical results for the Ising model are collected in an appendix.

1. PRELIMINARIES

In this section we precisely define the model and the random dynamics that will be the object of study in the next sections.

1.1. The Ising Model in a Finite Set

Let $\mathbf{Z}^2$ be the usual two-dimensional square lattice with sites $x = (x_1, x_2)$, equipped with the norm $\|x\| = |x_1| + |x_2|$. We will sometimes consider $\mathbf{Z}^2$ as a graph with vertices the sites $x \in \mathbf{Z}^2$ and edges all pairs of sites x and y such that $\|x-y\|=1$. We will use the notation σ to denote a generic element of the set $\Omega_{\mathbf{Z}^2}=\{-1,+1\}^{\mathbf{Z}^2}$; whenever $V\subset\mathbf{Z}^2$ we use the notation $\sigma_V=\{\sigma(x), x\in V\}$ to denote the restriction of σ to the set V and Ω_V to denote the set of them.

Given $V\subset\mathbf{Z}^2$, we define the interior and exterior boundaries of V as

$$\partial_{\text{int}} V\equiv\{x\in V;\ \exists y\notin V;\ \|x-y\|=1\}$$

$$\partial_{\text{ext}} V\equiv\{x\notin V; \exists y\in V;\ \|x-y\|=1\}$$

and the boundary ∂V as

$$\partial V=\{(x,y);\ x\in\partial_{\text{int}} V,\ y\in\partial_{\text{ext}} V;\ \|x-y\|=1\}$$

We also denote by $|V|$ the cardinality of V.

Next, for any finite subset V of $\mathbf{Z}^2$, we define the energy $H_V^{U^{\partial V},\tau}(\sigma_V)$ of a configuration $\sigma_V\in\Omega_V$ with boundary conditions τ outside V, $\tau\in\{-1,+1\}^{\mathbf{Z}^2}$, and boundary coupling $0\leqslant U^{\partial V}(x,y)\leqslant 1$, $(x,y)\in\partial V$, as

$$\begin{aligned}H_V^{U^{\partial V},\tau}(\sigma_V)=&-\tfrac{1}{2}\sum_{\substack{x,\,y\in V\\ \|x-y\|=1}}[\sigma_V(x)\,\sigma_V(y)-1]\\ &-\sum_{(x,y)\in\partial V}U^{\partial V}(x,y)\,[\sigma_V(x)\,\tau(y)-1]\end{aligned}\tag{1.1}$$

and the associated Gibbs probability measure at inverse temperature β:

$$\mu_V^{U^{\partial V},\tau}(\sigma_V)=Z(V,U^{\partial V},\tau)^{-1}\exp[-\beta H_V^{U^{\partial V},\tau}(\sigma_V)]\tag{1.2}$$

where the partition function $Z(V,U^{\partial V},\tau)$ is given by

$$Z(V,U^{\partial V},\tau)=\sum_{\sigma_V}\exp[-\beta H_V^{U^{\partial V},\tau}(\sigma_V)]\tag{1.3}$$

If the boundary condition τ is the special configuration $\tau(x)=1$ $\forall x\in\mathbf{Z}^2$, then in all our notation the superscript τ will be replaced by a simple $+$. Notice that the -1 appearing in the definition of the energy $H_V^{U^{\partial V},\tau}(\sigma_V)$ fixes equal to zero the energy with plus boundary conditions of the configuration identically equal to plus one.

We also set, for any function $f\colon\Omega_V\to\mathbf{R}$,

$$\mu_V^{U^{\partial V},\tau}(f)=\sum_{\sigma_V}\mu_V^{U^{\partial V},\tau}(\sigma_V)\,f(\sigma_V)$$

Although for technical reasons it will be convenient to consider cases in which the boundary coupling $U^{\partial V}(x,y)$ does depend on x and y and it is, for example, equal to plus one along some parts of the external boundary of V and positive but very weak along some other parts of the boundary, the most typical choices of $U^{\partial V}$ will be either $U^{\partial V}$ identically equal to one, in which case the Gibbs measure (1.2) is the usual Ising model in the set V with τ boundary conditions, or $U^{\partial V}$ identically equal to zero, which corresponds to the Ising model with open boundary conditions. In both cases the (cumbersome notation $\mu_V^{U^{\partial V},\tau}$, $Z(V, U^{\partial V}, \tau)$ will be replaced by the more natural ones μ_V^τ, $\mu_V^\varnothing$, $Z(V, \tau)$, $Z(V, \varnothing)$, respectively.

As a next step we recall some monotonicity properties enjoyed by the Gibbs measure $\mu_V^{U^{\partial V},\tau}$, which easily follow from the well-known FKG inequalities, (7) which will play a crucial role in the next sections.

Given two configurations τ_1, τ_2 in $\Omega_{\mathbf{Z}^2}$, we say that $\tau_1 \leqslant \tau_2$ iff

$$\tau_1(x) \leqslant \tau_2(x) \qquad \forall x \in \mathbf{Z}^2$$

and similarly for $\sigma_V, \sigma'_V \in \Omega_V$. Then, for any pair of finite subsets $V_1 \subset V_2$, any pair of boundary couplings $U_1^{\partial V_1}(x, y)$, $U_2^{\partial V_1}(x, y)$, and boundary conditions τ_1, τ_2 such that

$$U_1^{\partial V_1}(x, y)\, \tau_1(y) \leqslant U_2^{\partial V_1}(x, y)\, \tau_2(y) \quad \forall (x, y) \in \partial V_1$$

and any function $f\colon \Omega_{V_1} \to \mathbf{R}$ which is increasing with respect to the above partial order, we have

$$\mu_{V_1}^{U_1^{\partial V_1},\tau_1}(f) \leqslant \mu_{V_1}^{U_2^{\partial V_1},\tau_2}(f) \tag{1.4}$$

$$\mu_{V_2}^{U^{\partial V_2},\tau_1}(f) \leqslant \mu_{V_1}^{+}(f) \tag{1.5}$$

1.2. Contours and Cluster Expansion

In this section we recall, for the reader's convenience, a version of the cluster expansion for the partition function $Z(V, U^{\partial V}, +)$ valid under some restrictions on the boundary coupling $U^{\partial V}$, which will turn out to be quite essential in the next sections. The material that follows has been adapted to our situation, in which $U^{\partial V}$ is not necessarily identically equal to one, from Sections 3.8, and 3.9 of ref. 4.

To begin with, let us recall the definition of Peierls contours for a generic configuration σ which is identically equal to $+1$ outside a finite region.

If we denote by $\mathbf{Z}^{2*}$ the dual lattice of $\mathbf{Z}^2$, we call a *bond* any segment in $\mathbf{R}^2$ connecting two neighboring sites of $\mathbf{Z}^{2*}$. Then we say that two sites x and y in $\mathbf{Z}^2$ are separated by the bond h if their distance (as sites in $\mathbf{R}^2$) from h is equal to 1/2. Given $\sigma \in \Omega_{\mathbf{Z}^2}$, we denote by $\Gamma(\sigma)$ the collection of all bonds separating sites x and y in $\mathbf{Z}^2$ where $\sigma(x) \neq \sigma(y)$. If, moreover, we use the convention that any pair of orthogonal bonds that intersect in a given site x^* of the dual lattice $\mathbf{Z}^{2^*}$ are a *linked pair of bonds* iff they are both on the same side of the 45 deg line across x^*, then we immediately see that $\Gamma(\sigma)$ splits up in a unique way into a collection of closed contours $\Gamma_1(\sigma), \Gamma_2(\sigma),..., \Gamma_i(\sigma),...$, where a closed contour is a sequence $e_0, e_1, e_2,..., e_n$ of bonds such that:

(i) $e_i \neq e_j$ for all i and j with the exception of $i=0$ and $j=n$, for which $e_0 = e_n$.

(ii) For all i the bonds e_i and e_{i+1} have a common vertex in $\mathbf{Z}^{2^*}$.

(iii) If $e_i, e_{i+1}, e_j, e_{j+1}$ intersect at a given site x^*, then both e_i, e_{i+1} and e_j, e_{j+1} are linked pairs of bonds.

The length $|\Gamma|$ of a contour is simply the number of bonds in Γ. Given a contour Γ, we denote by $\Delta\Gamma$ the set of sites in $\mathbf{Z}^2$ such that either their distance (in $\mathbf{R}^2$) from Γ is 1/2 or their distance from the set of vertices of $\mathbf{Z}^{2^*}$ where two non-linked pair of bonds of Γ meet is equal to $1/\sqrt{2}$.

Since we can always identify any finite set $V \subset \mathbf{Z}^2$ with the bounded set $\tilde{V} \subset \mathbf{R}^2$ obtained by considering the union of all unit closed squares centered at each site in V, with an abuse of notation we will write for a generic closed contour Γ: $\Gamma \subset V$ if $\Gamma \subset \tilde{V}$ and $\Gamma \cap V$ for the set of bonds of $\Gamma(\sigma)$ contained in $\tilde{V}$.

Finally, given a boundary condition τ on the external boundary of a finite region V, we can associate to any element σ_V the configuration $\sigma^{(\tau+)} \in \Omega_{\mathbf{Z}^2}$ equal to σ_V inside V, equal to τ on $\partial_{\text{ext}} V$, and equal to $+1$ outside $V \cup \partial_{\text{ext}} V$. Then, via the previous construction, we can associate in a unique way to σ_V the *finite* collection of closed contours $\Gamma(\sigma^{(\tau+)})$ that, for simplicity, will be referred to as $\Gamma^\tau(\sigma_V)$. If we consider $\Gamma^\tau(\sigma_V) \cap V$, then it will consist of the union of some closed contours, in the sequel referred to as the closed contours of σ_V under the boundary condition τ, and some open polygonal curves that will be referred to as the open contours of σ_V under the boundary condition τ, where an open polygonal line is a sequence of distinct bonds $e_0, e_1, e_2,..., e_n$ satisfying (ii) and (iii) above.

Notice that, by construction, the first and last bond of an open contour necessarily separate at least one site in $\partial_{\text{int}} V$.

Let us now assume that

$$\alpha_V \equiv \min_{(x,y) \in \partial V} U^{\partial V}(x, y) > 0 \tag{1.6}$$

Then, if for a given closed contour Γ we write $\Gamma^{\partial V}$ for the sets of bonds in Γ that separate two sites $(x, y) \in \partial V$ and we set $U^{\partial V}(h) \equiv U^{\partial V}(x, y)$ for any pair $(x, y) \in \partial V$ that are separated by $h \in \Gamma^{\partial V}$, a simple computation shows that

$$H_V^{U^{\partial V}, +}(\sigma_V) = 2 \sum_{\Gamma \in \Gamma^+(\sigma_V)} \left\{ |\Gamma| - \sum_{h \in \Gamma^{\partial V}} [1 - U^{\partial V}(h)] \right\} \tag{1.7}$$

Thus the partition function can be written as

$$Z(V, U^{\partial V}, +) = \sum_{\sigma_V} \exp\left[-\beta \left(2 \sum_{\Gamma \in \Gamma^+(\sigma_V)} \left\{ |\Gamma| - \sum_{h \in \Gamma^{\partial V}} [1 - U^{\partial V}(h)] \right\} \right) \right] \tag{1.8}$$

We can rewrite (1.8) in a more suitable form by introducing the notion of compatibility between different contours.

We say that the contours $\Gamma_1, ..., \Gamma_n$ in V are *compatible* if there exists $\sigma_V \in \Omega_V$ such that $\Gamma^+(\sigma_V) = \{\Gamma_1, ..., \Gamma_n\}$ and we denote by $\mathscr{C}_V$ the set of them. Then, if we denote by $z_V^{U^{\partial V}}(\Gamma)$ the weight of a single contour Γ,

$$z_V^{U^{\partial V}}(\Gamma) = \exp\left(-\beta 2 \left\{ |\Gamma| - \sum_{h \in \Gamma^{\partial V}} [1 - U^{\partial V}(h)] \right\} \right) \tag{1.9}$$

we can write (1.8) as

$$Z(V, U^{\partial V}, +) = \sum_{\mathscr{G} \in \mathscr{C}_V} \prod_{\Gamma \in \mathscr{G}} z_V^{U^{\partial V}}(\Gamma) \tag{1.10}$$

Then the main result of the cluster expansion that is needed in the present paper can be stated as follows:

Proposition 1.1. Assume that there exists a constant $\alpha \in [0, 1)$ such that

$$z_V^{U^{\partial V}}(\Gamma) \leqslant \exp[-2\beta|\Gamma| \, (1 - \alpha)] \quad \forall \Gamma \in \mathscr{C}_V$$

Then there exists $\beta_0 = \beta_0(\alpha)$ such that for all $\beta \geqslant \beta_0$ the logarithm of the partition function $Z(V, U^{\partial V}, +)$ can be written as

$$\log[Z(V, U^{\partial V}, +)] = \sum_{\Lambda \subset V} \Phi^{U^{\partial V}, +}(\Lambda)$$

where the coefficients $\Phi^{U^{\partial V}, +}(\Lambda)$ satisfy the following two basic properties:
(1)

$$\Phi^{U^{\partial V}, +}(\Lambda) = \Phi^+(\Lambda) \qquad \text{if} \qquad \partial_{\text{ext}} \Lambda \subset V$$

where $\Phi^{+}(\Lambda)$ is the coefficient associated to the set $\Lambda \subset V$ when the boundary coupling $U^{\partial V}$ is identically equal to one; and (2) for all $\Lambda \subset V$, $|\Lambda| \geq 2$,

$$|\Phi^{U^{\partial V},+}(\Lambda)| \leqslant \exp\{-2(1-\alpha)\,[\beta-\beta_0]\,d(\Lambda)\}$$
$$|\Phi^{U^{\partial V},+}(\{x\})| \leqslant \exp\{-8(1-\alpha)\,[\beta-\beta_0]\}$$

where, for all connected (in the sense of subgraphs of the graph $\mathbf{Z}^2$) $\Lambda \subset V$, $d(\Lambda)$ is the length of the smallest connected set of bonds from $\bar{\Lambda} \equiv \{$all bonds in V that separate at least one site in $\Lambda\}$ containing all the bonds separating sites in $\partial_{\text{int}}\Lambda$ from sites in $\partial_{\text{ext}}\Lambda$. If Λ is not connected, $d(\Lambda) = +\infty$.

1.3. Surface Tension and Wulff Shape

We conclude our short review of the Ising model by recalling the definition of the surface tension $\tau_\beta(\mathbf{n})$ and of the associated Wulff shape. Again we follow the basic reference [4].

Let us fix a direction $\mathbf{n} \in S^1$ (S^1 being the unit circle) and let us define the boundary condition $\tau^{\mathbf{n}}$ as follows:

$$t^{\mathbf{n}}(x) = +1 \qquad \text{if} \quad (x, \mathbf{n}) > 0$$
$$t^{\mathbf{n}}(x) = -1 \qquad \text{otherwise}$$

where $(x, \mathbf{n})$ denotes the usual scalar product in $\mathbf{R}^2$.

Let also $V_{N,M}$ be the rectangle $\{x \in \mathbf{Z}^2;\ -N \leqslant x_1 \leqslant N;\ -M \leqslant x_2 \leqslant M\}$. Then we define the surface tension with respect to a surface orthogonal to the direction $\mathbf{n}$, $\tau_\beta(\mathbf{n})$, as

$$\tau_\beta(\mathbf{n}) \equiv \lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{\beta d(N, \mathbf{n})} \log\left(\frac{Z(V_{N,M}, t^{\mathbf{n}})}{Z(V_{N,M}, +)}\right) \tag{1.11}$$

where $d(N, \mathbf{n})$ is the length of the segment

$$\{x;\ (x, \mathbf{n}) = 0,\ -N \leqslant x_1 \leqslant N\}$$

We will simply write τ_β to denote the surface tension associated to the direction $\mathbf{n} = (1, 0)$.

For a proof of the existence of the limit (1.11) when β is large enough see Theorem 1.15 in ref. 4.

We now define the Wulff shape $W \subset \mathbf{R}^2$ as

$$W = \{x \in \mathbf{R}^2;\ |(x, \mathbf{n})| \leqslant \lambda \tau_\beta(\mathbf{n})\ \forall \mathbf{n}\} \tag{1.12}$$

where the constant λ is chosen in such a way that the area of W is equal to 1. The following fundamental result has been proved in ref. 4 (see also ref. 22):

Theorem 1.1. Let for any closed, piecewise smooth curve γ in $\mathbf{R}^2$, the Wulff functional $W_\tau(\gamma)$ on γ be given by

$$W_\tau(\gamma) = \int_\gamma ds\, \tau_\beta(\mathbf{b}(s))$$

where $\mathbf{n}(s)$ is the normal vector at the point s on the curve γ. Then, if we denote by ∂W the closed curve encircling the Wulff shape W, we have

$$W_\tau(\gamma) \geqslant W_\tau(\partial W)$$

for any closed curve γ which encloses an area equal to one, and equality holds iff γ is a translate of the curve ∂W.

1.3. A Class of Block-Glauber Dynamics for the Ising Model

In this section we define, for a given finite set $V \subset \mathbf{Z}^2$, boundary condition $\tau \in \Omega_{\mathbf{Z}^2}$, and boundary coupling $U^{\partial V}$, a class of Markov processes on Ω_V which are all reversible with respect to the Gibbs measure $\mu_V^{U^{\partial V}, \tau}$.

Although the main object of study in this work is any standard (e.g., Metropolis or heath bath *single-spin-flip* Markov process, reversible with respect to the Gibbs measure of the Ising model, we found it very convenient to introduce, as a technical tool, auxiliary Markov processes for which, in each updating of the dynamics, a whole collection of dynamical variables [i.e., spins $\sigma_V(x)$] are changed instead of just one. Each one of these auxiliary Markov processes will be indexed by a certain covering of the set V by *blocks* (i.e., subset of V) and at a given updating only the spins inside a particular block will be changed.

More precisely, let $\{Q_i\}_{i=1\ldots n}$ be a covering of V and let

$$\begin{aligned} U^{\partial Q_i}(x, y) &= 1 && \text{if} \quad (x, y) \in \partial Q_i \backslash \partial V \\ U^{\partial Q_i}(x, y) &= U^{\partial V}(x, y) && \text{if} \quad (x, y) \in \partial Q_i \cap \partial V \end{aligned} \tag{1.13}$$

Then we define the generator $L^{\{Q_i\}, \tau, U^{\partial V}}$ of the Markov process $\sigma_t^{\{Q_i\}, \tau, U^{\partial V}}$ indexed by the covering $\{Q_i\}_{i=1\ldots n}$ by

$$(L^{\{Q_i\}, \tau, U^{\partial V}} f)(\sigma_V) = \sum_i \sum_{\eta \in \Omega_{Q_i}} \mu_{Q_i}^{U^{\partial Q_i}, (\tau\sigma_V)}(\eta)\, [f(\sigma_V^\eta) - f(\sigma_V)] \tag{1.14}$$

where $(\tau\sigma_V)$ denotes the configuration in $\Omega_{\mathbf{Z}^2}$ equal to τ outside V and to σ_V inside V, while σ_V^η is the configuration in Ω_V equal to η in Q_i and to

$\sigma_{V\setminus Q_i}$ in $V\setminus Q_i$. Most of the time we will refer to the Markov process generated by $L^{\{Q_i\},\tau,U^{\partial V}}$ as the $\{Q_i\}$-dynamics.

A concrete way to construct the $\{Q_i\}$-dynamics starting from a configuration $\sigma\equiv\sigma_V$ is to choose with rate n (n is the cardinality of the covering) a particular element Q_i of the covering and to replace the restriction to Q_i of the configuration σ with a configuration $\eta\in\Omega_{Q_i}$ with probability $\mu_{Q_i}^{U^{\partial Q_i,(\tau\sigma)}}(\eta)$.

The particular case in which the elements Q_i of the covering are the sites x of the set V is known in the literature as the *heat bath process* (HB-dynamics in the sequel) and it is a particular example of a Glauber dynamics for the Ising model, that is, a Markov process on Ω_V with generator $L^{\tau,U^{\partial V}}$ of the form

$$(L^{\tau,U^{\partial V}}f)(\sigma_V)=\sum_{x\in V}\sum_{a\in\{-1,+1\}}c_x^{\tau,U^{\partial V}}(\sigma_V,a)\,[f(\sigma_V^{x,a})-f(\sigma_V)] \tag{1.15}$$

where $\sigma_V^{x,a}$ is obtained from σ_V by substituting the value $\sigma_V(x)$ with a and the jump rates $c_x^{\tau,U^{\partial V}}(\sigma_V,a)$ satisfy the *detailed balance condition*

$$\mu_V^{U^{\partial V},\tau}(\sigma_V)\,c_x^{\tau,U^{\partial V}}(\sigma_V,a)=\mu_V^{U^{\partial V},\tau}(\sigma_V^{x,a})\,c_x^{\tau,U^{\partial V}}(\sigma_V^{x,a},\sigma_V(x)) \tag{1.16}$$

and a short-range condition

$$c_x^{\tau,U^{\partial V}}(\sigma_V,a)=c_x^{\tau,U^{\partial V}}(\eta_V,a)\qquad\text{if}\quad\sigma_V(y)=\eta_V(y)\quad\forall\|x-y\|\leqslant R$$

for some finite R.

As it is easy to check, the $\{Q_i\}$-dynamics is a (continuous-time) Markov chain on Ω_V, reversible with respect to the Gibbs measure $\mu_V^{U^{\partial V},\tau}$; in other words, $L^{\{Q_i\},\tau,U^{\partial V}}$ is symmetric in the Hilbert space $L^2(\Omega_V,d\mu_V^{U^{\partial V},\tau})$ with real nonpositive eigenvalues

$$\begin{aligned}0=\lambda_0(\{Q_i\},\tau,U^{\partial V})>-\lambda_1(\{Q_i\},\tau,U^{\partial V})\geqslant\cdots\\ \geqslant-\lambda_k(\{Q_i\},\tau,U^{\partial V});\qquad k=2^{|V|}\end{aligned}$$

The absolute value of the first negative eigenvalue, $\lambda_1(\{Q_i\},\tau,U^{\partial V})$, will be of special value for us and it will be denoted by $\mathrm{gap}_V(\{Q_i\},\tau,U^{\partial V})$ or by $\mathrm{gap}_V(HB,\tau,U^{\partial V})$ if the dynamics under consideration is the heat-bath dynamics.

The following variational characterization of the gap will be particularly useful in the sequel. Let, for any $f\in L^2(\Omega_V,d\mu_V^{U^{\partial V},\tau})$, $\mathscr{E}(f,f)$ be the Dirichlet form associated to the generator $L^{\{Q_i\},\tau,U^{\partial V}}$:

$$\mathscr{E}(f,f)=\tfrac{1}{2}\sum_i\sum_{\sigma_V}\sum_{\eta\in\Omega_{Q_i}}\mu_V^{U^{\partial V},\tau}(\sigma_V)\,\mu_{Q_i}^{U^{\partial Q_i,(\tau\sigma_V)}}(\eta)\,[f(\sigma_V^{\eta})-f(\sigma_V)]^2 \tag{1.17}$$

Then

$$\text{gap}_V(\{Q_i\}) = \inf_{f \in L^2(\Omega_V, d\mu_V^{U^{\partial V}, \tau})} \frac{\mathscr{E}(f, f)}{\text{Var}(f)} \tag{1.18}$$

where

$$\text{Var}(f) = \tfrac{1}{2} \sum_{\sigma, \eta} \mu_V^{U^{\partial V}, \tau}(\sigma)\, \mu_V^{U^{\partial V}, \tau}(\eta)\, [f(\sigma) - f(\eta)]^2$$

Remark. Using the above variational characterization of the gap, it is very easy to check that, if we consider a general Glauber dynamics defined as in (1.15) with jump rates bounded above and below uniformly in σ_V and in V, then the corresponding gap can be bounded from above and from below by $\text{gap}_V(HB, \tau, U^{\partial V})$ multiplied by two suitable constants.

The following simple estimate, which follows from an elementary L^2 consideration, illustrates the role played by the $\text{gap}(\{Q_i\}, \tau, U^{\partial V})$ in the approach to the invariant measure $\mu_V^{U^{\partial V}, \tau}$ of the distribution $P_{V, \eta_V}^{\{Q_i\}, \tau, U^{\partial V}}(t)$ of the $\{Q_i\}$-dynamics at time t starting from η_V at time $t = 0$:

$$\| P_{V, \eta_V}^{\{Q_i\}, \tau, U^{\partial V}} - \mu_V^{U^{\partial V}, \tau} \| \leqslant \frac{\exp[-t\, \text{gap}(\{Q_i\}, \tau, U^{\partial V})]}{2[\mu_V^{U^{\partial V}, \tau}(\eta_V)]^{1/2}} \tag{1.19}$$

where, for two arbitrary probability measures ν and μ on Ω_V, $\|\nu - \mu\|$ denotes their variation distance.

Remark. It is worthwhile to observe that (1.19) can be a very bad estimate since the denominator $[\mu_V^{U^{\partial V}, \tau}(\eta_V)]^{1/2}$ is of order $\exp(-c\beta |V|)$ for some constant c. There are situations, for example, when β is smaller than the critical value β_c, in which the factor $[\mu_V^{U^{\partial V}, \tau}(\eta_V)]^{-1/2}$ in (1.19) can be replaced by $c|V|$ for some constant c (see, e.g., refs. 28, 18, and 19). However, in a phase transition regime, $\beta > \beta_c$, the gap can be very small, something like $\exp(-cL)$ if V is a square of side L with open boundary conditions (see Section 4), and therefore the possible improvement in the denominator from $\exp(-c\beta |V|)$ to some negative power of $|V|$ is negligible.

1.4. Coupling for the $\{Q_i\}$ Dynamics

We conclude this preparatory section by discussing a useful coupling for the $\{Q_i\}$-dynamics that will be essential in the forthcoming sections.

Let, for any finite set V, $\tau^{(1)}, \tau^{(2)}, \ldots, t^{(N)}$ be $N \leqslant 2^{|\partial_{\text{ext}} V|}$ boundary conditions on the external boundary of V, and let $\nu_V^{\tau^{(1)}, \tau^{(2)}, \ldots, t^{(N)}}$ be the unique invariant probability measure on $(\Omega_V)^N$ (N copies of Ω_V) of the following ergodic Markov process:

(i) With rate $|V|$ one chooses a site $x \in V$ and, given x, a random number $\xi_x \in [0, 1]$ with a uniform distribution:

$$ \tag{1.20} $$

(ii) For $k = 1,..., N$ the value of the spin at x in the kth component of the initial configuration $\tilde{\sigma}_V \equiv \{\sigma_V^{(1)} \cdots \sigma_V^{(N)}\}$, $\sigma_V^{(k)} \in \Omega_V$, is replaced by $+1$ if

$$\xi_x \leqslant \mu_{\{x\}}^{U^{\partial\{x\}}, (\tau^{(k)}\sigma_V^{(k)})}(+1) \tag{1.21}$$

and by -1 if the opposite inequality holds. Here $U^{\partial\{x\}}$ is defined as in (1.13) but with Q_i replaced by $\{x\}$.

The above algorithm is of course nothing more than an explicit way to realize on a common probability space the HB-dynamics in V with different boundary conditions $\tau^{(1)}, \tau^{(2)},..., \tau^{(N)}$.

Using this observation, one can explicitly check that the measure $\nu_V^{\tau^{(1)}, \tau^{(2)},..., t^{(N)}}$ enjoys the following properties:

$$\sum_{\substack{\eta^{(1)},..., \eta^{(k-1)} \\ \eta^{(k+1)},..., \eta^{(N)}}} \nu_V^{\tau^{(1)}, \tau^{(2)},..., \tau^{(N)}}(\eta^{(1)},..., \eta^{(k-1)}, \eta^{(k)}, \eta^{(k+1)},..., \eta^{(N)}) = \mu_V^{U^{\partial V}, (\tau^{(k)})}(\eta^{(k)}) \tag{1.22}$$

$$\nu_V^{\tau^{(1)}, \tau^{(2)}, \tau^{(N)}}(\eta^{(k)} \leqslant \eta^{(j)}) = 1 \qquad \text{if} \quad \tau^{(k)} \leqslant \tau^{(j)} \tag{1.23}$$

Given now a finite set V, a boundary condition τ, and a covering $\{Q_i\}_{i=1}^n$, let, for each $i = 1,..., n$, $\tau^{(1)}, \tau^{(2)},..., t^{(N)}$ be an arbitrary enumeration of all the possible boundary conditions on the external boundary of Q_i which agree with τ on $\partial_{\text{ext}} Q_i \cap \partial_{\text{ext}} V$, and let

$$\nu_{Q_i} \equiv \nu_{Q_i}^{\tau^{(1)}, \tau^{(2)},..., t^{(N)}}$$

Using the measures ν_{Q_i}, we can mimick the algorithm (1.20), (1.21), to realize on a common probability space the Markov processes $\sigma_t^{\{Q_i\}, \tau, U^{\partial V}}$ starting from an arbitrary initial condition σ_V as follows:

(a) With rate n (n is the cardinality of the covering) we choose one of the Q_i.

$$ \tag{1.24} $$

(b) For all $k = 1,..., N$, the configurations σ_V which agree with $\tau^{(k)}$ on the external boundary of Q_i are updated to $\sigma_V^{\eta^{(k)}}$, $\eta^{(k)} \in \Omega_{Q_i}$, and the joint probability of $\eta^{(1)}, \eta^{(2)},..., \eta^{(N)}$ is $\nu_{Q_i}(\eta^{(1)}, \eta^{(2)},..., \eta^{(N)})$:

$$ \tag{1.25} $$

It is clear that, because of (1.22) above, (a) and (b) give the right law for the evolution of any given initial configuration σ_V. Moreover, because of (1.23), it also follows that any ordered set of initial conditions $\sigma^1_V \leqslant \sigma^2_V \leqslant \cdots \leqslant \sigma^k_V$ will remain ordered for any future time t. We will refer to this last property as monotonicity in the initial configuration.

2. GEOMETRIC BOUNDS ON THE GAP

In this section we establish two basic estimates on the gap which, besides being interesting by themselves, will play a crucial role in the determination of the exact asymptotics in the thermodynamic limit of the gap of the HB-dynamics in a finite square with open boundary conditions. The first estimate relates $\text{gap}_V(\text{HB}, \tau, U^{\partial V})$ to $\text{gap}_V(\{Q_i\}, \tau, U^{\partial V})$ when V is a rectangle $V_{N,M}$,

$$V_{N,M} = \{x;\ -N \leqslant x_1 \leqslant N;\ -M \leqslant x_2 \leqslant M\}$$

with, say, $M \leqslant N$ and the covering $\{Q_i\}$ consists of rectangles

$$Q_i = \left\{x \in \mathbf{Z}^2;\ -N \leqslant x_1 \leqslant N;\ i\frac{l}{2} \leqslant x_2 \leqslant (i+2)\frac{l}{2}\right\}$$

with $l/2$ and $2M/l$ integers, $i = -2M/t, \ldots, 2M/l - 2$. The estimate shows that the ratio

$$\frac{\text{gap}_V(\text{HB}, \tau, U^{\partial V})}{\text{gap}_V(\{Q_i\}, \tau, U^{\partial V})}$$

is bounded from below by a suitable exponential of the *short* side l. More precisely:

Theorem 2.1. Let V and $\{Q_i\}$ be as above. Then for any boundary coupling $U^{\partial V}$ and any boundary condition τ we have

$$\text{gap}_V(\text{HB}, \tau, U^{\partial V}) \geqslant \frac{1}{2|Q_i|} \frac{\text{ex}(-4\beta)}{\exp(-4\beta) + \exp(+4\beta)} \times \exp[-4\beta(l+1)]\, \text{gap}_V(\{Q_i\}, \tau, U^{\partial V})$$

Remark. The above theorem remains valid also if the covering of the set V consists of rectangles Q_i with longest side smaller than that of $V_{N,M}$. However, for reasons that will become clear in the next section, the above choice of the covering is very sensible in the low-temperature regime. It will also become clear at the end of the proof of the theorem that our method allows one to relate the gap of the HB-dynamics to that of the $\{Q_i\}$-dynamics for arbitrary geometric shapes of the elements of the covering. This generality is, however, not needed in the present paper.

As a corollary we obtain, in the same setting as above, that $\text{gap}_V(\text{HB}, \tau, U^{\partial V})$ is not smaller than a negative exponential of the shortest side M. More precisely we have:

Corollary 2.1. For any boundary coupling $U^{\partial V}$ and any boundary condition τ we have

$$\text{gap}_V(\text{HB}, \tau, U^{\partial V}) \geqslant \frac{1}{2|V|} \frac{\exp(-4\beta)}{\exp(-4\beta)+\exp(+4\beta)} \exp[-4\beta(2M+1)]$$

Proof of the Corollary. Let us take in Theorem 2.1 the shortest side l of the elements Q_i of the covering equal to $2M$ so that the covering consists of just the rectangle $V_{N,M}$ itself. Then the generator $L^{\{Q_i\}, \tau, U^{\partial V}}$ restricted to the space of functions of mean zero (i.e., orthogonal to the constant functions) becomes minus the identity, so that $\text{gap}_V(\{Q_i\}, \tau, U^{\partial V}) = 1$, and the corollary follows from Theorem 2.1.

Remark. The estimate described in the corollary is a very bad one for temperatures above the critical one (that is, $\beta < \beta_c$), since in this case it has been recently proved by Martinelli *et al.*[20] that the gap is bounded away from zero uniformly in N and M. However, at low temperature, when the infinite-volume dynamics is not ergodic, it gives the right dependence on the size of the set $V_{N,M}$, namely a negative exponential of the surface and not of the volume $|V_{N,M}|$, but the constant in the exponential is wrong by a factor 2 even in the limit $\beta \to \infty$. A more precise bound will be discussed in the next section.

The proof of the corollary represents also the first, actually rather trivial, example of the role played by the $\{Q_i\}$-dynamics: in an approach to a gap estimate this latter may be considerably simpler than the single-spin dynamics. In particular, one may try to attack the problem of finding a lower bound on the gap of the HB-dynamics by first proving lower bounds on the gap of the $\{Q_i\}$-dynamics and then, using Theorem 2.1 above, transfer the bound to the HB-dynamics. This idea played an important role in the analysis of the approach to equilibrium in general Glauber dynamics in the one-phase region (see, e.g, refs. 8, 28, and 18). However, its application in the phase transition region seems to be new.

Proof of Theorem 2.1. The proof is an application in our context of some geometric techniques developed a few years ago in order to bound from below the gap of symmetric Markov chains on complicated graphs (see, e.g., refs. 14, 11, 12, 27, and 5). We will use in particular some beautiful ideas introduced by Jerrum and Sinclair in their study of rapid mixing properties of Markov chains arising in some hard computational problems. The way these techniques apply to spin dynamics like the Glauber dynamics was discussed for the first time in some unpublished notes of mine and used recently, in a slightly different form by Schonmann in his study of metastability for the Ising model.[24]

In what follows we will omit for simplicity in all the notation the boundary condition τ, the boundary coupling $U^{\partial V}$, and the volume V. Thus, for example, the Gibbs measure $\mu_V^{U^{\partial V},\tau}$ will become μ, the conditional Gibbs measure on Q_i, $\mu_{Q_i}^{U^{\partial Q_i,(\tau\sigma)_V}}(\eta)$, $\mu_{Q_i}^{\sigma}(\eta)$, and similarly for the generators of the HB- and $\{Q_i\}$-dynamics together with their gaps.

We start by introducing the set of *canonical paths* in Ω_V between configurations σ and σ' with $\sigma \neq \sigma'$, which are connected by just one single jump of the $\{Q_i\}$-dynamics, that is, $\sigma' = \sigma^\eta$ for some i and some $\eta \in \Omega_{Q_i}$. We adopt the convention that, if the σ, σ' can be connected by the updating either Q_i or Q_{i+1}, due to their mutual overlap, then we think of σ' as arising from the updating of Q_i.

Let us first order the sites in each rectangle Q_i as follows:

$$x < y \qquad \text{iff} \quad x_1 < y_1 \quad \text{or} \quad x_1 = y_1 \quad \text{and} \quad x_2 < y_2$$

Given now $\sigma \in \Omega_V$ and $\eta \in \Omega_{Q_i}$, we define the path $\gamma(\sigma, \sigma^\eta)$ as the sequence of configurations obtained from σ by adjusting one by one, in increasing order, the values of its spins in Q_i to those of the spins of σ^η. More precisely, if $x_1, ..., x_n$ are the sites in Q_i, ordered as above, such that $\sigma(x_i) \neq \sigma^\eta(x_i)$, then we define $\gamma(\sigma, \sigma^\eta) = \{\sigma^0 \cdots \sigma^n\}$, where σ^i, $i = 1, ..., n$, is the configuration equal to

$$\begin{aligned} \sigma^i(x) &= \sigma^\eta(x) \qquad \text{iff} \quad x \leqslant x_i \\ \sigma^i(x) &= \sigma(x) \qquad \text{iff} \quad x > x_i \end{aligned} \tag{2.1}$$

and $\sigma^0 = \sigma$.

Next, for any allowed transition of the HB-dynamics

$$\bar\sigma \to \bar\sigma^{x,a}, \qquad x \in Q_i, \quad a = -\bar\sigma(x)$$

we set

$$\begin{aligned} e &= (\bar\sigma, \bar\sigma^{x,a}) \\ Q(e) &= \mu(\bar\sigma)\, \mu_x^{\bar\sigma}(a) \end{aligned} \tag{2.2}$$

and we say that the transition e belongs to the canonical path γ, $e \in \gamma$ if, for some index i, $(\bar\sigma, \bar\sigma^{x,a}) = (\sigma^i, \sigma^{i+1})$. Finally, we define the constant ρ as

$$\rho = \sup_{i,e} \sum_{\substack{\sigma,\eta \\ e \in \gamma(\sigma,\sigma^\eta)}} \frac{\mu(\sigma)\, \mu_{Q_i}^{\sigma}(\eta)}{Q(e)} \tag{2.3}$$

Then we have:

$$\operatorname{gap}_V(HB) \geqslant \frac{1}{2|Q_i|}\frac{1}{\rho}\operatorname{gap}_V(\{Q_i\}) \tag{2.4}$$

Although the proof of (2.4) can be found in ref. 27, we reproduce it below because of its simplicity.

Using the variational principle for any $f \in L^2(\Omega_V, d\mu)$, we have

$$\begin{aligned}\operatorname{Var}(f) &\leqslant \operatorname{gap}_V(\{Q_i\})^{-1}\tfrac{1}{2}\sum_{\sigma,i}\sum_{\eta}\mu(\sigma)\,\mu^{\sigma}_{Q_i}(\eta)\,[f(\sigma^{\eta})-f(\sigma)]^2\\ &= \operatorname{gap}_V(\{Q_i\})^{-1}\tfrac{1}{2}\sum_{\sigma,i}\sum_{\eta}\mu(\sigma)\,\mu^{\sigma}_{Q_i}(\eta)\left\{\sum_{j=1\cdots n}[f(\sigma^{j})-f(\sigma^{j-1})]\right\}^2\end{aligned} \tag{2.5}$$

where $\gamma(\sigma, \sigma^{\eta}) = \{\sigma^0\sigma^1\cdots\sigma^{\eta}\}$ is the canonical path going from σ to σ^{η}.

Using the Schwartz inequality, the fact that the length n of the path is smaller than $|Q_i|$, and the definition of ρ, we can bound from above the r.h.s. of (2.5) by

$$\begin{aligned}&\operatorname{gap}_V(\{Q_i\})^{-1}\rho|Q_i|\tfrac{1}{2}\sum_{\sigma,i}\sum_{x\in Q_i}\sum_{a\in\{-1,1\}}\mu(\sigma)\,\mu^{\sigma}_x(a)\,[f(\sigma^{x,a})-f(\sigma)]^2\\ &\quad\leqslant 2\operatorname{gap}_V(\{Q_i\})^{-1}\rho|Q_i|\,\mathscr{E}_{\mathrm{HB}}(f,f)\end{aligned} \tag{2.6}$$

where $\mathscr{E}_{\mathrm{HB}}(f,f)$ is the Dirichlet form of the HB-dynamics and the factor 2 in the first inequality comes from the fact that most of the sites belong to two elements of the covering. Thus, if we combine (2.5) with (2.6) and (1.18), we get

$$\operatorname{gap}_V(\mathrm{HB}) \geqslant \frac{\operatorname{gap}_V(\{Q_i\})}{2\rho|Q_i|}$$

and in order prove the theorem, we only need to estimate from above the constant ρ by

$$\rho \leqslant \frac{\exp(4\beta)+\exp(-4\beta)}{\exp(-4\beta)}\exp[4\beta(l+1)] \tag{2.7}$$

uniformly in the boundary condition τ and in the bounary coupling $U^{\partial V}$.

Apparently this is not an easy problem since we have to count how many canonical paths use a given allowed transition $e = (\bar{\sigma}, \bar{\sigma}^{x,a})$. Is is precisely at this stage that Jerrum and Sinclair's lovely ideas become essential.

Given a transition $e = (\bar{\sigma}, \bar{\sigma}^{x,a})$, we define an *injective* Φ mapping from the set of all the canonical paths that use the transition e, $\Gamma(e)$, to Ω_V as follows:

$$\begin{aligned} &\Phi(\gamma(\sigma, \sigma''))\,(y) = \sigma(y) && \forall y \in Q_i, \quad y < x \\ &\Phi(\gamma(\sigma, \sigma''))\,(y) = \sigma''(y) && \forall y \in Q_i, \quad y \geqslant x \\ &\Phi(\gamma(\sigma, \sigma''))\,(y) = \sigma(y) && \forall y \notin Q_i \end{aligned} \tag{2.8}$$

where the index i labels the rectangle associated to the path $\gamma(\sigma, \sigma'')$.

It is clear that Φ is injective. In fact the knowledge of the transition e, that is, of x and $\bar{\sigma}$, *and* of $\xi \equiv \Phi(\gamma(\sigma, \sigma''))$ allow us to reconstruct completely the initial and final configurations σ and σ'' and thus the path itself, simply by observing that, for example,

$$\begin{aligned} &\sigma(y) = \bar{\sigma}(y) && \forall y \notin Q_i \\ &\sigma(y) = \bar{\sigma}(y) && \forall y \in Q_i, \quad y \geqslant x \\ &\sigma(y) = \xi(y) && \forall y \in Q_i, \quad y < x \end{aligned} \tag{2.9}$$

and similarly for σ''.

Let now c_0 be the smallest constant such that for any canonical path $\gamma(\sigma, \sigma'')$ in $\Gamma(e)$ the following bound holds:

$$\mu^{\sigma}_{Q_i}(\Phi(\gamma))\, Q(e) \geqslant \frac{1}{c_0}\, \mu(\sigma)\, \mu^{\sigma}_{Q_i}(\eta) \tag{2.10}$$

Then we have

$$\rho \leqslant c_0 \tag{2.11}$$

Using (2.10), we can in fact estimate the r.h.s. of (2.3) by

$$c_0 \sup_{e, i} \sum_{\gamma \in \Gamma(e)} \mu^{\sigma}_{Q_i}(\Phi(\gamma)) \tag{2.12}$$

Since the map Φ is injective and μ is a probability measure, the sum in (2.12) is not greater than one and (2.11) follows.

In order to estimate the constant c_0, let, for $x \in Q_i$, ∂_x be the set of bonds in Q_i which separates sites in Q_i smaller than or equal to x from sites in Q_i larger than x. Clearly, by construction, ∂_x consists of two vertical segments joined by a single horizontal bond h at distance 1/2 from x and placed below it if $x_1 \geqslant -N+1$, and by a single vertical segment plus a horizontal bond as above if $x_1 = -N$. Let also, for any pair configurations σ_1, σ_2 which agree outside Q_i, $H_{\partial_x}(\sigma_1, \sigma_2)$ be the interaction through

∂_x of a configuration $\sigma \in \Omega_V$ which is equal to σ_1 (σ_2) to the left (right) of ∂_x. More precisely

$$H_{\partial_x}(\sigma_1, \sigma_2) = - \sum_{\substack{y \leqslant x < z \\ x \in Q_l,\ \|y-z\|=1}} [\sigma_1(y)\,\sigma_2(z) - 1] \tag{2.13}$$

Clearly $|H_{\partial_x}(\sigma_1, \sigma_2)|$ is bounded from above by $2(l+1)$. Then, by direct inspection,

$$\frac{\mu(\sigma)\,\mu^{\sigma}_{Q_l}(\eta)}{\mu^{\sigma}_{Q_l}(\Phi(\gamma))\,\mu(\bar{\sigma})} = \exp\{-\beta[H_{\partial_x}(\sigma, \sigma) + H_{\partial_x}(\sigma'', \sigma'') - H_{\partial_x}(\bar{\sigma}, \bar{\sigma}) - H_{\partial_x}(\Phi(\gamma), \Phi(\gamma))]\} \tag{2.14}$$

for any boundary condition τ and boundary coupling $U^{\partial V}$. In turn (2.14), together with (2.13) and the observation that

$$\mu^{\bar{\sigma}}_x(a) \geqslant \frac{\exp(-4\beta)}{\exp(-4\beta) + \exp(+4\beta)} \qquad \forall x, \bar{\sigma}, \tau, U^{\partial V}$$

implies that the l.h.s. of (2.14) is smaller than

$$\frac{\exp(4\beta) + \exp(-4\beta)}{\exp(-4\beta)} \exp[4\beta(l+1)]\,\mu^{\bar{\sigma}}_x(a) \tag{2.15}$$

that is,

$$\mu^{\sigma}_{Q_l}(\Phi(\gamma))\,Q(e) \geqslant \frac{\exp(4\beta) + \exp(-4\beta)}{\exp(-4\beta)} \exp[4\beta(l+1)]\,\mu(\sigma)\,\mu^{\sigma}_{Q_l}(\eta) \tag{2.16}$$

Thus the constant c_0 can be taken equal to

$$c_0 = \frac{\exp(4\beta) + \exp(-4\beta)}{\exp(-4\beta)} \exp[4\beta(l+1)]$$

Using (2.7) and, (2.11), the theorem follows.

Remark. It is amusing to observe that, if one applies the above construction to the one-dimensional case for which the set ∂_x consists of just *one* bond, $H_{\partial_x}(\sigma_1, \sigma_2)$ can be bounded by a constant independent of σ_1, σ_2 and of the dimension of V, even if the energy (1.1) of a configuration σ_V is replaced by a more general expression like

$$H(\sigma_V) = -\tfrac{1}{2} \sum_{x, y \in V} J(\|x-y\|)\,\sigma_V(x)\,\sigma_V(y) + \text{b.c.}$$

provided that the long-range potential $J(\|x-y\|)$ decays faster than $\|x-y\|^{-2+\varepsilon}$ for some $\varepsilon>0$. Therefore in this case the gap of the corresponding heat bath dynamics in a segment of length L in $\mathbf{Z}$ has a lower bound which is only proportional to L^{-1} without any negative exponential of L.

On the other hand, it is known that a long-range potential $J(\|x-y\|)$ with a fast decay as above is not able to induce any phase transition, the reason being that the energy between two semiinfinite lines is finite uniformly in the spin configuration.

Thus, in some sense, the above geometric construction is able to capture, at least at the level of the exponential, some (but certainly not all) of the physical aspects of the presence (or of the absence) of a phase transition in the Ising model at low temperature.

3. A LOWER BOUND ON THE GAP WITH PLUS BOUNDARY CONDITIONS AND ITS APPLICATION

In this section we consider the HB-dynamics in a square $V=V_L$:

$$V_L=\{x\in\mathbf{Z}^2;\ 0\leqslant x_i\leqslant L;\ i=1,2\}$$

with full plus boundary conditions, that is,

$$\tau(x)=+1 \qquad \forall x\in\mathbf{Z}^2$$

$$U^{\partial V_L}(x,y)=+1 \qquad \forall (x,y)\in\partial V_L$$

and very large β.

We show that, due precisely to the presence of the plus boundary conditions, the gap of the HB-dynamics is much larger, as $L\to\infty$, than its value with *open* boundary conditions (see also the discussion in the introduction). As a simple consequence, we show that the equal site time correlations of the infinite-volume process started in the plus phase decay faster than any inverse power of the time.

Before stating and discussing our main result, let us fix a few more convenient notations. We will denote by $(+)$ and $(-)$ the two extreme configurations in Ω_{V_L} identically equal to plus and minus one, respectively, and, for any rectangle R, by $\mu_R^{\tau_1,\tau_2,\tau_3,\tau_4}$, the Gibbs measure on R with the boundary conditions τ_1, τ_2, τ_3, τ_4 on the external boundary of its four sides ordered clockwise starting from the bottom side. We use the usual convention that, if one of the configurations τ_i is identically equal to $+1$ or -1, then we replace it by a plus or a minus sign. Thus, for example, τ_1, $+$, $-$, $+$ means τ_1 boundary conditions on the bottom side, plus boundary conditions on the vertical ones, and minus boundary condition on the top one. Whenever confusion does not arise we will also omit the subscript V in the notation σ_V.

We finally denote by μ^+ the infinite-volume Gibbs state obtained as the limit as $L \to \infty$ of finite-volume Gibbs states μ_V^+ with plus boundary conditions, by $m^*(\beta) = \mu^+(\sigma(0))$ the spontaneous magnetization, and by σ_t the infinite-volume heat bath dynamics started from the (infinite-volume) configuration σ (see ref. 13 for the existence of such processes).

We can now state the main results:

Theorem 3.1. Let $\varepsilon \in (0, 1/2)$ be given. Then there exist $\beta_0 < +\infty$ and $C < +\infty$ such that for any $\beta \geqslant \beta_0$ and any integer L

$$\mathrm{gap}_{V_L}(\mathrm{HB}, +) \geqslant \exp(-C\beta L^{1/2+\varepsilon})$$

Theorem 3.2. Let $\alpha \in [0, 2)$ be given. Then there exist $\beta_0 < +\infty$ and $C < +\infty$ such that for any $\beta \geqslant \beta_0$

$$0 \leqslant \int d\mu^+(\sigma)\,\sigma(0)\,E(\sigma_t(0)) - [m^*(\beta)]^2$$

$$\leqslant C\exp\{-(\log(t))^\alpha\} \quad \forall t$$

Proof of Theorem 3.1. Let $l = 2[L^{1/2+\varepsilon}]$ and let us suppose, without loss of generality, that $N \equiv 2L/l - 1$ is an integer; for $i = 1, ..., N$, we define Q_i to be the rectangle

$$Q_i = \left\{x \in V_L; 0 \leqslant x_1 \leqslant L,\ (i-1)\frac{l}{2} \leqslant x_2 \leqslant (i+1)\frac{l}{2}\right\}$$

Then, using Theorem 2.1, we have that

$$\mathrm{gap}_{V_L}(\mathrm{HB}, +) \geqslant \frac{1}{|Q_i|}\frac{\exp(-4\beta)}{\exp(-4\beta) + \exp(+4\beta)}$$
$$\times \exp[-4\beta(l+1)]\,\mathrm{gap}_{V_L}(\{Q_i\}, +) \qquad (3.1)$$

It remains to show that the $\{Q_i\}$-dynamics has a "large" gap, where "large" means, for example, larger than $\exp(-L^{(1+\varepsilon)/2})$.

To prove this result, we will show that, with very large probability, under the coupling for the $\{Q_i\}$-dynamics described at the end of Section 1, the two extreme configurations $(+)$ and $(-)$ become identical in a time smaller than $\exp(L^{(1+\varepsilon)/2})$.

The intuitive reason for that, which also explains our apparently strange choice of the length l of the short side of Q_i, is the following. Let us suppose that we start with the two extreme configurations and that we update one after the other in increasing order of i the rectangles Q_i. In the first updating of Q_1 we have to replace $(+)_{Q_1}$ and $(-)_{Q_1}$ with two configurations $\eta^+_{Q_1}$ and $\eta^-_{Q_1}$ distributed according to $\mu^{+,+,+,+}_{Q_1}$ and $\mu^{+,+,-,+}_{Q_1}$, respectively. It is a relatively easy matter to show (see Proposition 3.1 below) that, for large enough β, due to our choice of l and to the fact that in two dimensions the fluctuations of an interface separating plus spins from minus spins are of the order of the *square root* of the length of the interface, it is possible to couple the two measures $\mu^{+,+,+,+}_{Q_1}$, $\mu^{+,+,-,+}_{Q_1}$ in such a way that, with probability much larger than $1-1/N$, the two configurations $\eta^+_{Q_1}$ and $\eta^-_{Q_1}$ are identical in a large portion of Q_1, e.g., for all $x \in Q_1$ with $x_2 \leqslant 3l/4$, and in particular on the external boundary of the bottom side of Q_2. Moreover, with large probability, both $\eta^+_{Q_1}$ and $\eta^-_{Q_1}$ will be mostly $+1$ on the external boundary of the bottom side of Q_2. Thus the second updating in Q_2 will be very similar to the first one in Q_1 with the exception that now the boundary conditions on the external boundary of the bottom side of Q_2 will not be identically equal to $+$ but only approximately.

As we will show below, this fact, with probability much larger than $1-1/N$, does not really matter and one can, at least in a first approximation, consider the $+$ boundary conditions also on the bottom side of Q_2. In this approximation the second updating will be statistically equal to the first one and, with large probability, it will force $(+)^{\eta^+_{Q_1}}$ and $(-)^{\eta^-_{Q_1}}$ to agree also in 3/4 of Q_2 without introducing any new discrepancy between them in the previous region of agreement,

$$\left\{x \in Q_1;\ x_2 \leqslant \frac{3l}{4}\right\}$$

In such a way, after the first two updatings, the evolutes of $(+)$ and $(-)$ will agree in the set

$$\{x \in V_L;\ 0 \leqslant x_2 \leqslant \tfrac{5}{4} l\}$$

By iterating this procedure N times, we can glue together $(+)$ and $(-)$ in N steps with a probability of order one.

Remark. Thus our choice of the short side l is a compromize between the requirement of being as small as possible because of Theorem 2.1 and the requirement of being much larger than the *square root* of L, which is the order of magnitude of the typical fluctuations of an interface of length L in two dimensions.

Since the probability of *not* having within time t a sequence of N updatings exactly in the order needed above is roughly of order

$$\exp\left(-N^{-N}\frac{t}{N}\right) \ll 1 \qquad \text{if, e.g.,} \quad t=\exp(L^{(1+\varepsilon)/2}), \quad L \gg 1$$

we can conclude that the time the $\{Q_i\}$-dynamics needs to relax to equilibrium should not be larger than $\exp(L^{(1+\varepsilon)/2})$ for L large enough.

Let us start with the technicalities. Let R be the rectangle

$$R=\{x\in \mathbf{Z}^2;\ 0\leqslant x_1\leqslant L_1;\ 0\leqslant x_2\leqslant L_2\}$$

with $L_1\geqslant L_2\geqslant L_1^{1/2+\varepsilon}$.

Proposition 3.1. Let $m>0$ and $\varepsilon\in(0, 1/2)$ be given. Then there exists $\beta_0\equiv\beta_0(\varepsilon, m)$ independent of R such that for all $\beta\geqslant\beta_0$ and all $x\in R$ with $x_2\leqslant 3/4L_2$, we have

$$\mu_R^{+,+,+,+}(\sigma(x)=1)-\mu_R^{+,+,-,+}(\sigma(x)=1)\leqslant \exp(-mL_1^{2\varepsilon})$$

The above result will actually be given in a greater generality than that required here; see Proposition 4.1. The proof of Proposition 4.1 has been collected with some similar results for the Ising model in the Appendix.

The second result that we need is an estimate on the probability of not seeing within time t a sequence of updatings of the $\{Q_i\}$-dynamics with the correct order described above.

Lemma 3.1. Let us call $S_N\equiv\{t_1,..., t_N\}$, $N=2L/l-1$, an ordered sequence of updatings if for any $i=1,..., N$: (i) at time t_i the dynamics updates the rectangle Q_i; (ii) there are no updatings between times t_i and t_{i+1}.

Then, for any N large enough (independent of t)

$$P(\text{there exists no ordered sequence in } [0, t])\leqslant \exp\left(-\frac{tN^{-N}}{2}\right)$$

Proof. Given that $t_1,..., t_N$ are N consecutive updatings, the probability that $S_N\equiv\{t_1,..., t_N\}$ is an ordered sequence is clearly N^{-N} since the probability of choosing a specific rectangle is $1/N$. Let now ν_t denote the total number of updatings within time t. By construction the process ν_t is a Poisson process of parameter tN. Therefore we can estimate the probability appearing in the lemma by

$$P(\text{there exists no ordered sequence in } [0, t])$$
$$\leqslant \sum_{k=0}^{+\infty} \frac{e^{-tN}(tN)^k}{k!}(1-N^{-N})^{[k/N]}$$
$$\leqslant 2e^{-tN[1-(1-N^{-N})^{1/N}]}$$

which is smaller, for N large enough, than

$$\exp\left(-\frac{tN^{-N}}{2}\right) \tag{3.3}$$

Let now $S_N \equiv \{t_1,..., t_N\}$ be a fixed ordered sequence with $t_1 = 0$, let $\sigma_{t_i}^{\{Q_i\},+}$ be the evolute at time t_i of the initial configuration σ, let R_i be the rectangle

$$R_i = \left\{x \in \bigcup_{j \leqslant i} Q_j;\ x_2 \leqslant (i+1)\frac{l}{2} - \left[\frac{l}{4}\right]\right\}$$

and let, for $i = 1,..., N-1$, $A_i(x)$, A_i, be the events

$$A_i(x) = \{(+)_{t_i}^{\{Q_i\},+}(x) \neq (-)_{t_i}^{\{Q_i\},+}(x)\}$$
$$A_i = \bigcup_{\{x \in R_i\}} A_i(x) \tag{3.4}$$
$$A_N = \bigcup_{\{x \in V_L\}} A_N(x)$$

and let $q_i = P(A_i)$. Then we have

$$q_{n+1} \leqslant q_n + P(A_{n+1} \cap A_n^c)$$
$$\leqslant \sum_{n=1}^{N-1} P(A_{n+1} \cap A_n^c) + P(A_1) \tag{3.5}$$

where A_n^c is the complement set of A_n.

Then the term $P(A_{n+1} \cap A_n^c)$ in the r.h.s. of (3.5) can be estimated by

$$P(A_{n+1} \cap A_n^c)$$
$$\leqslant \sum_{\substack{x \in R_{n+1} \cap Q_{n+1} \\ \sigma \in \Omega_V}} \mu_V^+(\sigma)\, P\Bigg(A_{n+1}(x)$$
$$\cap \left[\bigcap_{y \in R_n} \{(+)_{t_n}^{\{Q_i\},+}(y) = (-)_{t_n}^{\{Q_i\},+}(y) = \sigma_{t_n}^{\{Q_i\},+}(y)\}\right]\Bigg) \tag{3.6}$$

In the derivation of (3.6) we used the fact that at time t_{n+1} we update only the set Q_{n+1} and that, under the coupling described in Section 1.4, for any time t and any configuration σ, $(-)_t^{\{Q_i\},+} \leqslant \sigma_t^{\{Q_i\},+} \leqslant (+)_t^{\{Q_i\},+}$.

In turn, if we denote by E the expectation over the random configuration $\sigma_{t_n}^{\{Q_i\},+}$, then a given term in the sum appearing in the r.h.s. of (3.6) can be estimated from above by

$$\begin{aligned} \mu_V^+(\sigma)\, E[\mu_{Q_{n+1}}^{\sigma_{t_n}^{\{Q_i\},+},+,(+)_{t_n}^{\{Q_i\},+},+}&(\eta(x)=1) \\ &- \mu_{Q_{n+1}}^{\sigma_{t_n}^{\{Q_i\},+},+,(-)_{t_n}^{\{Q_i\},+},+}(\eta(x)=1)] \\ = [\mu_V^+(\sigma)\, E\mu_{Q_{n+1}}^{\sigma_{t_n}^{\{Q_i\},+},+,+,+}&(\eta(x)=1) \\ &- \mu_V^+(\sigma)\, E\mu_{Q_{n+1}}^{\sigma_{t_n}^{\{Q_i\},+},+,-,+}(\eta(x)=1)] \end{aligned} \tag{3.7}$$

since $(+)_{t_n}^{\{Q_i\},+}$ and $(-)_{t_n}^{\{Q_i\},+}$ are, respectively, identically equal to plus one and minus one on the external boundary of the top of Q_{n+1} because the sequence S_N is ordered.

Let us consider the term

$$\sum_{\sigma \in \Omega_V} \mu_V^+(\sigma)\, E\mu_{Q_{n+1}}^{\sigma_{t_n}^{\{Q_i\},+},+,+,+}(\eta(x)=1) \tag{3.8}$$

Since the $\{Q_i\}$-dynamics is reversible with respect to $\mu_V^+(\sigma)$, the distribution of $\sigma_{t_n}^{\{Q_i\},+}$, given that σ is distributed according to $\mu_V^+(\sigma)$, will of course be again $\mu_V^+(\sigma)$. Therefore (3.8) will be equal to

$$\begin{aligned} &\sum_{\sigma \in \Omega_V} \mu_V^+(\sigma)\, \mu_{Q_{n+1}}^{\sigma,+,+,+}(\eta(x)=1) \\ &\qquad \leqslant \sum_{\sigma \in \Omega_{R_{n+1} \cup Q_{n+1}}} \mu_{R_{n+1} \cup Q_{n+1}}^{+,+,+,+}(\sigma)\, \mu_{Q_{n+1}}^{\sigma,+,+,+}(\eta(x)=1) \end{aligned} \tag{3.9}$$

where we used once more the monotonicity (1.5).

By the DLR property of the Gibbs measure $\mu_{R_{n+1} \cup Q_{n+1}}^{+,+,+,+}$, the r.h.s. of (3.9) is just

$$\mu_{R_{n+1} \cup Q_{n+1}}^{+,+,+,+}(\sigma(x)=1) \tag{3.10}$$

Similarly we obtain that the term

$$\sum_{\sigma \in \Omega_V} \mu_V^+(\sigma)\, E\mu_{Q_{n+1}}^{\sigma_{t_n}^{\{Q_i\},+},+,-,+}(\eta(x)=1)$$

is bounded from below by

$$\mu^{+,+,-,+}_{R_{n+1}\cup Q_{n+1}}(Q(x)=1) \tag{3.11}$$

In conclusion, using (3.10), (3.11), and Proposition (3.1), we get that, for any n, the r.h.s. of (3.6) is bounded from above by

$$\mu^{+,+,+,+}_{R_{n+1}\cup Q_{n+1}}(\sigma(x)=1)-\mu^{+,+,-,+}_{R_{n+1}\cup Q_{n+1}}(\sigma(x)=1)\leqslant L^2\exp(-mL^{2\varepsilon}) \tag{3.12}$$

for a suitable constant $m\equiv m(\beta)$ which diverges as $\beta\to\infty$. Similarly one estimates $P(A_1)$.

Therefore we get

$$q_N\leqslant NL^2\exp(-mL^{2\varepsilon}) \tag{3.13}$$

We are now in a position to conclude the proof of the theorem.

Given a sequence $S_N\equiv\{t_1,..., t_N\}$ of updatings, we say that S_N is a *good* sequence iff S_N is ordered and the event A^c_N occurred at the end of the sequence. Because of (3.13) we know that the probability that an ordered sequence is also a good sequence is larger than

$$1-NL^2\exp(-mL^{2\varepsilon})>\tfrac{1}{2}$$

for L large enough. Thus, using Lemma 3.1, we get that if $T=\exp(L^{(1+\varepsilon)/2})$ and L is large enough,

$$P(\text{there exists a good sequence in } [0, T])\geqslant\tfrac{1}{3} \tag{3.14}$$

We conclude by observation that, if there exists a good sequence in $[0, t]$, then, by monotonicity (see Section 1.4), the evolutes at the end of the sequence of $(+)$ and of $(-)$ will be identical. Therefore we can estimate $P((+)^{\{Q_i\},+}_t\neq(-)^{\{Q_i\},+}_t)$ by

$$P((+)^{\{Q_i\},+}_t\neq(-)^{\{Q_i\},+}_t)\leqslant(\tfrac{2}{3})^{[t/T]} \tag{3.15}$$

which immediately implies that

$$\text{gap}_{V_L}(\{Q_i\}, +)\geqslant T^{-1}\log(\tfrac{3}{2})=\exp(-L^{(1+\varepsilon)/2})\log(\tfrac{3}{2}) \tag{3.16}$$

Clearly (3.16) together with (3.1) proves the theorem.

Proof of Theorem 3.2. The first inequality, namely

$$0\leqslant\int d\mu^+(\sigma)\,\sigma(0)\,E(\sigma_t(0))-[m^*(\beta)]^2 \tag{3.17}$$

follows immediately from the FKG inequality applied to μ^+ and the fact that the infinite-volume heat bath dynamics is reversible with respect to μ^+.

In order to obtain the upper bound, we write the r.h.s. as

$$\int d\mu^+(\sigma)\,[\sigma(0)+1]\,E(\sigma_t(0))-[m^*(\beta)]^2-m^*(\beta) \tag{3.18}$$

and we observe that, by the monotonicity (1.24), (1.25), for any L and any t,

$$E(\sigma_t(0))\leqslant E^+_{V_L,+}(\sigma_t(0)) \tag{3.19}$$

where $E^+_{V_L,+}$ denotes the expectation over the HB-dynamics in V_L with plus boundary conditions starting from the configuration identically equal to plus one.

In turn, the r.h.s. of (3.19) can be bounded above, using the estimate (1.19), by

$$E^+_{V_L,+}(\sigma_t(0))\leqslant \mu^+_{V_L}(\sigma(0))+\exp[C\beta L^2-t\,\text{gap}(\text{HB},V_L,+)] \tag{3.20}$$

If we plug (3.19) into (3.18) and we use (3.20), we obtain that the r.h.s. of (3.17) is bounded above by

$$[\mu^+_{V_L}(\sigma(0))-m^*(\beta)]\,[m^*(\beta)+1]+2\exp[C\beta L^2-t\,\text{gap}(\text{HB},V_L,+)] \tag{3.21}$$

As is well known,

$$0\leqslant \mu^+_{V_L}(\sigma(0))-m^*(\beta)\leqslant C_1\exp(-mL) \tag{3.22}$$

for any large enough β, where C_1 and m are suitable constants with $m\to\infty$ as $\beta\to\infty$.

We now choose the size L depending on t as

$$L=\left[\frac{\log(t)}{2C(\alpha)\beta}\right]^\alpha \tag{3.23}$$

where $C(\alpha)$ is the constant appearing in Theorem 3.1 for the value $\varepsilon=(2-\alpha)/2\alpha$ and we apply Theorem 3.1 to get that the r.h.s. of (3.21) is bounded from above by

$$C_1\exp\left(-m\left[\frac{\log(t)}{2C(\alpha)\beta}\right]^\alpha\right)+C_2\exp\left(-\frac{\sqrt{(t)}}{2}\right) \tag{3.24}$$

for all β large enough, where C_2 is a suitable constant.

Clearly (3.24) proves the theorem.

4. ASYMPTOTICS OF THE GAP WITH OPEN BOUNDARY CONDITIONS

In this section we again consider the HB-dynamics in a square $V \equiv V_L$ of side L at very low temperature, but this time with open boundary conditions, that is,

$$U^{\partial V_L}(x, y) = 0 \quad \forall (x, y) \in \partial V_L$$

In this case the two extremal configurations, $(+)$ and $(-)$, are the only absolute minima of the energy $H_V^{\varnothing}(\sigma_V)$ and they are related one to the other by a global spin flip.

We show that, due precisely to the above symmetry, the gap of the HB-dynamics is much smaller, as $L \to \infty$, than its value with plus boundary conditions. More precisely, we obtain that the gap is of the order of $\exp(-\beta\tau_\beta L)$, where τ_β is the surface tension defined in (1.11) with respect to an interface parallel to one of the coordinate axes.

Since the proof of the main result of the present section (see Theorem 4.1 below) will mimick as closely as possible the proof of Theorem 3.1, we will keep the same notation as Section 3 with the following modification.

Let R be a rectangle and let us suppose that we have a boundary coupling $U^{\partial R}$ which is constant on each of the four components of ∂R ordered clockwise starting from the bottom. Let us denote by $0 \leqslant \delta_i \leqslant 1$, $i = 1,..., 4$, the value of the boundary coupling on the ith side of R. Then we will write $\mu_R^{\delta_1\tau_1, \delta_2\tau_2, \delta_3\tau_3, \delta_4, \tau_4}$, to denote the corresponding Gibbs measure on R with the boundary conditions $\tau_1, \tau_2, \tau_3, \tau_4$. As usual, if one the δ_i is equal to one, it will be omitted in the notation, while if it is zero, the corresponding term $\delta_i\tau_i$ will be replaced by $\varnothing$. Thus, for example, $(\tau_1, \delta+, \varnothing, \delta+)$ means τ_1 boundary conditions on the bottom side, plus boundary conditions on the vertical ones coupled to the interior of R by a constant boundary coupling equal to δ, and an open boundary condition on the top one.

As in Section 3, whenever confusion does not arise, we will omit the subscript V in the notation σ_V.

Let us now state the main result:

Theorem 4.1. Let $\varepsilon \in (0, 1/4)$ be given. Then there exist $\beta_0 < +\infty$ and $C < +\infty$ such that for any $\beta \geqslant \beta_0$ and any integer L

$$\exp(-\beta\tau_\beta L - C\beta L^{1/2+\varepsilon}) \leqslant \text{gap}_{V_L}(\text{HB}, \varnothing) \leqslant \exp(-\beta\tau_\beta L + C\beta L^{1/2+\varepsilon})$$

Proof. Upper Bound. The idea behind the upper bound is very simple and intuitive: when the system starts from a typical configuration of the Gibbs measure $\mu_V^\varnothing$ it has a magnetization m approximately equal to either $+m^*(\beta)$ or $-m^*(\beta)$, where $m^*(\beta)$ is the value of the spontaneous magnetization at inverse temperature β in the infinite-volume limit. Therefore, in order to reach the equilibrium where the expected value of the magnetization is zero by symmetry, the process has to hit the set of configurations of zero magnetization. Since the probability that, starting at equilibrium, one has at a given time t zero magnetization is equal to $\mu_V^\varnothing(m=0)$, one expects the relaxation time to equilibrium, which is roughly the inverse of the gap, to be at least as large as the inverse of $\mu_V^\varnothing(m=0)$. This is actually correct and the argument, thanks to a basic result of Shlosman (see Theorem 4.2 below), gives a correct upper bound.

Let us implement the above idea. Without a true loss of generality we may assume that L^2 is odd. We also denote by $m(\sigma)$ the total magnetization of the configuration $\sigma \in \Omega_V$:

$$m(\sigma) = \sum_{x \in V} \sigma(x)$$

and by $\langle \cdot,\cdot \rangle$ the scalar product in $L^2(\Omega_V, d\mu_V^\varnothing)$.

If we recall that the generator of the dynamics $L^\varnothing$ is self-adjoint on $L^2(\Omega_V, d\mu_V^\varnothing)$, we get that

$$\begin{aligned}\langle m; \exp(tL^\varnothing) m \rangle &\\ &\leqslant \exp[-\text{gap}_{V_L}(\text{HB}, \varnothing) t]\, \langle m; m \rangle \\ &\leqslant L^4 \exp[-\text{gap}_{V_L}(\text{HB}, \varnothing) t] \end{aligned} \tag{4.1}$$

since, by symmetry, $\langle m; 1 \rangle = 0$.

On the other hand, again by symmetry,

$$[\exp(tL^\varnothing) m]\,(\sigma) = -[\exp(tL^\varnothing) m]\,(-\sigma) \tag{4.2}$$

so that

$$\langle m; \exp(tL^\varnothing) m \rangle = 2 \int_{\sigma:\, m(\sigma) > 0} d\mu_V^\varnothing(\sigma)\, m(\sigma)\, [\exp(tL^\varnothing) m]\,(\sigma) \tag{4.3}$$

If we denote by $T^{(m<0)}(\sigma)$ the first hitting time of the set $\{m(\sigma) < 0\}$ for the HB-dynamics in V starting at time $t = 0$ from the configuration σ, we get that, for configurations σ with positive magnetization, $[\exp(tL^\varnothing) m]\,(\sigma)$ can bounded from below by

$$\begin{aligned}[\exp(tL^\varnothing) m]\,(\sigma) &\geqslant P(T^{(m<0)}(\sigma) > t) - L^2 P(T^{(m<0)}(\sigma) \leqslant t) \\ &= 1 - (L^2 + 1)\, P(T^{(m<0)}(\sigma) \leqslant t)\end{aligned} \tag{4.4}$$

since $\inf_{\sigma:\, m(\sigma) \geqslant 0} m(\sigma) = 1$ in view of our condition that L^2 is odd.

A rather standard computation in the theory of Glauber dynamics that uses the invariance of the measure $\mu_V^{\varnothing}(\sigma)$ and the fact that

$$P(\nu_t \geqslant 2L^2 t) \leqslant \exp(-KL^2 t)$$

for a suitable constant K, where ν_t is the number of updatings within time t, shows that

$$\sum_{\sigma} \mu_V^{\varnothing}(\sigma)\, P(T^{(m<0)}(\sigma) \leqslant t) \leqslant 2L^2 t \mu_V^{\varnothing}(m(\sigma)=1) + \exp(-KL^2 t) \tag{4.5}$$

If we insert (4.4) and (4.5) in (4.3), we get that

$$\langle m; \exp(tL^{\varnothing}) m \rangle \geqslant 2\mu_V^{\varnothing}(m(\sigma) \geqslant 0) - 4(L^2+1) \times L^2 t \mu_V^{\varnothing}(m(\sigma)=1) - 2(L^2+1)\exp(-KL^2 t) \tag{4.6}$$

By symmetry $\mu_V^{\varnothing}(m(\sigma) \geqslant 0) = 1/2$, so that, for all L large enough and all

$$1 \leqslant t \leqslant [16(L^2+1)\, L^2 \mu_V^{\varnothing}(m(\sigma)=1)]^{-1} \tag{4.7}$$

the r.h.s. of (4.6) is greater than 1/4.

If we combine this result with (4.1), we obtain

$$\tfrac{1}{4} \leqslant L^4 \exp(-\mathrm{gap}_{V_L}(\mathrm{HB}, \varnothing)\, t) \quad \forall 1 \leqslant t \leqslant [16(L^2+1)\, L^2 \mu_V^{\varnothing}(m(\sigma)=1)]^{-1} \tag{4.8}$$

We use at this point a fundamental result due to Shlosman (see Theorem 3 in ref. 26) in his study of the Wulff shape in a finite square with periodic or open boundary conditions:

Theorem 4.2 (Shlosman). There exists β_0 such that for any $\beta \geqslant \beta_0$ and any sequence of integers ρ_L, $L \in \mathbf{N}$, satisfying

$$\lim_{L \to \infty} \frac{\rho_L}{L^2} = \rho \in (-m^*(\beta), m^*(\beta)), \qquad \rho_L - L^2 = \bmod 2$$

the limit

$$\psi(\rho) = \lim_{L \to \infty} -\frac{1}{\beta L} \log[\mu_V^{\varnothing}(m(\sigma) = \rho_L)]$$

exists and is given by

$$\psi(\rho) = \frac{1}{2} w \left(\frac{m^*(\beta) - |\rho|}{2m^*(\beta)} \right)^{1/2}, \qquad |\rho| \geqslant \rho_1,$$

$$\psi(\rho) = \frac{1}{2} \omega \left(\frac{m^*(\beta) - \rho_1}{2m^*(\beta)} \right)^{1/2}, \qquad |\rho| \leqslant \rho_1$$

where the constant w is the value of the Wulff functional W_τ on the Wulff curve ∂W (see Theorem 1.1) and the singularity point ρ_1 satisfies the equation

$$\frac{1}{2} w \left(\frac{m^*(\beta)-|\rho_1|}{2m^*(\beta)}\right)^{1/2} = \tau_\beta$$

Warning. Due to some misprints, the formula for $\psi(\rho)$ in ref. 26 appears with 1/2 and $[(m^*(\beta)-|\rho|)/2m^*(\beta)]^{1/2}$ replaced by 1/4 and $[m^*(\beta)-|\rho|]^{1/2}$, respectively.

Remark.. Given $\varepsilon \in (0, 1/4)$ and β large enough, it is possible to show, using the methods of ref. 4, that the above limit is approached, as $L \to \infty$, at least as fast as $L^{-1/2+\varepsilon}$.

By plugging in (4.8) the result of Theorem 4.2 and its strengthening mentioned in the remark above, we immediately obtain the required upper bound on the gap.

Remark. Actually the above reasoning leads to an upper bound on the gap which is a negative exponential of the surface in *any* dimension $d \geqslant 2$ if we use the estimate of Schonmann (25)

$$\mu_V^{\varnothing}(m(\sigma)=0) \leqslant \exp[-c(\beta)\, L^{d-1}]$$

for a suitable constant c. Moreover, it is possible to show[2] that in two dimensions the above estimate is valid for *any* β larger than the critical value β_c. Therefore, using Corollary 2.1 and the above observation, we get that in $d=2$ for any $\beta > \beta_c$ there exist two constants c_1 and c_2 such that for any L large enough

$$\exp(-c_1 L) \leqslant \text{gap}_{V_L}(\text{HB}, \varnothing) \leqslant \exp(-c_2 L)$$

It would be nice to show that at least one of the two constants is equal to $\beta\tau_\beta$.

We finally notice that it was possible to follow a slightly different proof by using in the variational characterization of the gap the trial function

$$f(\sigma) = \chi(m(\sigma) > 0) - \chi(m(\sigma) < 0)$$

$\chi(A)$ being the characteristic function of the event A, and then exploiting Shlosman's result.

Lower Bound. We start by replacing the open boundary conditions on ∂V by very weak *plus* boundary conditions. More precisely, let

$$\delta = L^{-1/2}$$

and let us consider a constant boundary coupling $U^{\partial V}(x, y)$:

$$U^{\partial V}(x, y) = \delta \quad \forall (x, y) \in \partial V$$

Then, in the notation established in Section 1.3 and at the beginning of the present section, we trivially have for any $a \in \{-1, +1\}$

$$\begin{aligned} \exp(-8\beta\, \delta L)\, \mu_V^{\delta+,\delta+,\delta+,\delta+}(\sigma) \leqslant \mu_V^{\varnothing}(\sigma) \leqslant \exp(8\beta\, \delta L)\, \mu_V^{\delta+,\delta+,\delta+,\delta+}(\sigma) \\ \exp(-8\beta\, \delta L)\, \mu_{\{x\}}^{U^{\partial x},(\sigma,\delta+)}(a) \leqslant \mu_{\{x\}}^{(\sigma,\varnothing)}(a) \leqslant \exp(8\beta\, \delta L)\, \mu_{\{x\}}^{U^{\partial x},(\sigma,\delta+)}(a) \end{aligned} \tag{4.9}$$

where $\mu_{\{x\}}^{(\sigma,\varnothing)}$ is the conditional probability of having the value a for $\sigma(x)$ given that outside V there are open boundary conditions and that the configuration in $V \backslash \{x\}$ is σ. Similarly for $\mu_{\{x\}}^{U^{\partial x},(\sigma,\delta+)}(a)$.

It is immediate to check, using the variational characterization of the gap in terms of the Dirichlet form (1.17), that (4.9) implies the following bound on $\text{gap}_{V_L}(\text{HB}, \varnothing)$ in terms of $\text{gap}_{V_L}(\text{HB}, +, \delta)$:

$$\text{gap}_{V_L}(\text{HB}, \varnothing) \geqslant \exp(-32\beta\, \delta L)\, \text{gap}_{V_L}(\text{HB}, +, \delta) \tag{4.10}$$

It is therefore sufficient to establish the correct lower bound with "$\delta+$" boundary conditions.

To this purpose we proceed exactly as in Section 3, namely we consider the $\{Q_i\}$-dynamics with Q_i as in the proof of Theorem 3.1 and estimate $\text{gap}_{V_L}(\text{HB}, +, \delta)$ by

$$\begin{aligned} \text{gap}_{V_L}(\text{HB}, +, \delta) \geqslant \frac{1}{|Q_i|} \frac{\exp(-4\beta)}{\exp(-4\beta) + \exp(+4\beta)} \\ \times \exp[-4\beta(l+1)]\, \text{gap}_{V_L}(\{Q_i\}, +, \delta) \end{aligned} \tag{4.11}$$

where $l = 2[L^{1/2+\varepsilon}]$.

The main difference now with the reasoning behind the proof of Theorem 3.1 is the following. When we start from the two extremal configurations $(+)$ and $(-)$ at the beginning of an ordered sequence S_N and we update the first rectangle Q_1, we replace $(+)_{Q_1}$ and $(-)_{Q_1}$ with two configurations $\eta^+_{Q_1}$ and $\eta^-_{Q_1}$ distributed according to $\mu_{Q_1}^{\delta+,\delta+,+,\delta+}$ and $\mu_{Q_1}^{\delta+,\delta+,-,\delta+}$, respectively. Contrary to the "full" (i.e., $\delta = 1$) plus boundary conditions discussed in Theorem 3.1, the measure $\mu_{Q_1}^{\delta+,\delta+,-,\delta+}$ is *not* concentrated for β large on configurations which resemble those of the plus phase, at least far from the top side, but instead, due precisely to the "full" minus boundary condition on the top side, on configurations in which the spins are mostly minus one with little islands of plus spins. Therefore, with large probability, the first updating of the ordered sequences will *not* force $(+)$ and $(-)$ to agree in a large portion (e.g., 3/4) of Q_1.

We notice, however, that with very small probability the two new configurations $(+)^{\eta^+_{Q_1}}$ and $(-)^{\eta^-_{Q_1}}$ *will* agree in, say, 3/4 of Q_1 if, for example, the interface in the configuration $\eta^-_{Q_1}$ separating the minus spins on the top side of Q_1 from the plus spins in the rest of the boundary instead of being in its typical position, namely close to the bottom side of Q_1, is very close to the top one. It turns out that the probability in question is at least of the order of $\exp(-\beta\tau_\beta L)$. Once this rare event has occurred, then, in the second updating, we will have to consider the Gibbs measures $\mu_{Q_2}^{\eta^+_{Q_1},\delta+,+,\delta+}$ and $\mu_{Q_2}^{\eta^+_{Q_1},\delta+,-,\delta+}$, which, if we approximate , as we did in the introduction to the proof of Theorem 3.1, the boundary condition $\eta^+_{Q_1}$ with a "full" plus, become $\mu_{Q_1}^{+,\delta+,+,\delta+}$ and $\mu_{Q_1}^{+,\delta+,-,\delta+}$.

Now the situation is very different from the first updating and much more similar to the case treated in the proof of Theorem 3.1. In fact, in the Gibbs measure $\mu_{Q_1}^{+,\delta+,-,\delta+}$, the "full" plus boundary condition on the bottom side compensate, exactly the "full" minus boundary condition on the top one and therefore the "phase" (that is, the structure of the typical configurations) is decided by the lateral "$\delta+$" boundary conditions. Since the typical fluctuations of the interface separating the minus spin of the top from the plus spins at the bottom are of order $\sqrt{L} \ll l$, and since $\delta l = L^\varepsilon \gg 1$, one can conclude (see Proposition 4.1 below) that the above two Gibbs measures are very similar in, say, 3/4 of Q_1. Thus the second updating will, with large probability, enlarge the region of agreement between the evolutes of $(+)$ and $(-)$ to

$$\{x \in V,\ 0 \leqslant x_2 \leqslant \tfrac{5}{4} l\}$$

Iterating this procedure, we see that an ordered sequence $S_N = \{t_1, ..., \tau_N\}$ will typically glue together $(+)$ and $(-)$ with the last updating at time t_N, provided that in the first one, at time t_1, a very rare event of probability of order $\exp(-\beta\tau_\beta L)$ has occurred.

Clearly the above reasoning implies that the relaxation time to equilibrium for the $\{Q_i\}$-dynamics should be at most of order $\exp(+\beta\tau_\beta L)$ and therefore, using (4.11), the required lower bound would follow.

Let us implement the above program. We start by giving a generalization to the case of $\delta+$ lateral boundary conditions of Proposition 3.1. As in Section 3, let R be a rectangle

$$R=\{x\in\mathbf{Z}^2;\ 0\leqslant x_1\leqslant L_1,\ 0\leqslant x_2\leqslant L_2\}$$

with $L_1\geqslant L_2\geqslant L_1^{1/2+\varepsilon}$. Then we have:

Proposition 4.1. Let $m>0$ and $\varepsilon\in(0,1/2)$ be given and let $\delta=L_1^{-1/2}$. Then there exists $\beta_0\equiv\beta_0(\varepsilon,m)$ such that for all $\beta\geqslant\beta_0$ and all $x=(x_1,x_2)\in R$ with $x_2\leqslant\frac{3}{4}L_2$ we have

$$\mu_R^{+,\delta+,+,\delta+}(\sigma(x)=1)-\mu_R^{+,\delta+,-,\delta+}(\sigma(x)=1)\leqslant\exp(-mL_1^{\varepsilon})$$

Moreover, if R and R' are two rectangles as above with the same basis L_1 but different heights $L_1\geqslant L_2\geqslant L_2'\geqslant L_1^{1/2+\varepsilon}$, then for all $x=(x_1,x_2)\in R$ with, for example, $x_2\leqslant\frac{1}{16}L_2'$, we have

$$\mu_{R'}^{\delta+,\delta+,+,\delta+}(\sigma(x)=1)-\mu_R^{\delta+,\delta+,+,\delta+}(\sigma(x)=1)\leqslant\exp(-mL_1^{1/2+\varepsilon})$$

For a proof see the appendix.

There is an interesting corollary to the above proposition that can be viewed as a generalization of Theorem 3.1 to the case when we have open boundary conditions on three sides of the square V_L and full plus boundary conditions on the remaining one.

Corollary 4.1. Let $\varepsilon\in(0,1/2)$ be given. Then there exist $\beta_0<+\infty$ and $C<+\infty$ such that for any $\beta>\beta_0$ and any integer L

$$\mathrm{gap}_{V_L}(\mathrm{HB},\varnothing,\varnothing,+,\varnothing)\geqslant\exp(-C\beta L^{1/2+\varepsilon})$$

Proof. We use (4.10) to replace the open boundary conditions on the three sides by $\delta+$ boundary conditions. Then we can repeat word for word the proof of Theorem 3.1, with Proposition 3.1 replaced by Proposition 4.1.

The second new result that we need is as follows.

For a given rectangle R as above and $\sigma\in\Omega_R$, let $\Gamma^{+,+,-,+}(\sigma)$ be the family of contours of σ with boundary condition τ having the constant signs $+$, $+$, $-$, $+$ on the external boundary of the four sides of R ordered in the usual way. As one can immediately check, under the above boundary conditions there exists only one open contour, which will be denoted by $\Gamma_{R,\,\mathrm{open}}^{+,+,-,+}(\sigma)$.

We then define the event $\mathscr{A}_R^{+,+,-,+}$ as

$$\mathscr{A}_R^{+,+,-,+}=\left\{\sigma;\ \Gamma_{R,\,\mathrm{open}}^{+,+,-,+}(\sigma)\subset\left\{x\in R;\ x_2>\frac{13L_2}{16}\right\}\right\}\qquad(4.12)$$

Proposition 4.2. In the hypotheses of Proposition 4.1 there exists a positive constant C independent of β and L_1 such that

$$\mu_R^{\delta+,\delta+,-,\delta+}(\mathscr{A}_R^{+,+,-,+}) \geqslant \exp(-\beta\tau_\beta L_1 - C\beta L_1^\varepsilon)$$

For a proof see the appendix.

We are now in a position to complete the proof of the lower bound.

As a first step and for reasons that will appear clear later in the proof, it is convenient to modify slightly the coupling for the Q_i-dynamics. More precisely, we use the same algorithm described in (1.24), (1.25), but with a modified coupled measure $\tilde{\nu}_{Q_1}$ for the first rectangle Q_1. The measure $\tilde{\nu}_{Q_1}$, which will be obtained from the old one ν_{Q_1} via "surgery" (see, for instance, ref. 3) on a suitable subset of Q_1, will, however, still enjoy the monotonicity property described at the end of Section 1.

Let $\tilde{R}_1$ be the rectangle

$$\tilde{R}_1 = \left\{x \in Q_1;\ x_2 \leqslant l - \frac{3l}{16}\right\}$$

and let $\nu_{Q_1}^{\tau^{(1)}\cdots\tau^{(N)}}$ be the measure on $(\Omega_{Q_1})^N$, $N = 2^L$, constructed in Section 1.4, with boundary conditions $\delta+$ on the bottom and lateral sides of Q_1 and $\tau^{(1)},..., \tau^{(N)}$ on the top side of Q_1.

We then construct the new measure $\tilde{\nu}_{Q_1}$ on $\Omega_{Q_1}^N$ as follows.

Given N configurations $\sigma^{(1)},..., \sigma^{(N)}$ in $\Omega_{\mathscr{L}}$, where $\mathscr{L}$ denotes the external boundary of the top side of $\tilde{R}_1$, let $\nu_{Q_1\setminus\mathscr{L}}^{\sigma^{(1)}\cdots\sigma^{(N)};\tau^{(1)}\cdots\tau^{(N)}}$ be the measure constructed according to (1.20), (1.21) for the set $Q_1\setminus\mathscr{L}$ and boundary conditions:

- $\delta+$ on the bottom and lateral sides of Q_1.
- $\sigma^{(1)},..., \sigma^{(N)}$ on $\mathscr{L}$.
- $\tau^{(1)},..., \tau^{(N)}$ on the top side of Q_1, where $\tau^{(1)},..., \tau^{(N)}$ are *all* possible configurations on the external boundary of the top side of Q_1.

It is very important to notice that $\nu_{Q_1\setminus\mathscr{L}}^{\sigma^{(1)}\cdots\sigma^{(N)};\,\tau^{(1)}\cdots\tau^{(N)}}$ is a product of the measures

$$\nu_{\tilde{R}_1}^{\sigma^{(1)}\cdots\sigma^{(N)}} \qquad \text{and} \qquad \nu_{Q_1\setminus\{\tilde{R}_1\cup\mathscr{L}\}}^{\sigma^{(1)}\cdots\sigma^{(N)};\,\tau^{(1)}\cdots\tau^{(N)}}$$

where, for notational convenience, we have omitted to indicate the fixed $\delta+$ boundary conditions on the bottom and lateral sides of Q_1.

Finally, given N configurations $\tilde{\sigma}^{(1)},..., \tilde{\sigma}^{(N)}$ in Ω_{Q_1}, we set

$$\begin{aligned}\tilde{\nu}_{Q_1}^{\tau^{(1)}\cdots\tau^{(N)}}&(\tilde{\sigma}^{(1)},..., \tilde{\sigma}^{(N)})\\ &= \sum_{\sigma^{(1)}\cdots\sigma^{(N)}} \nu_{Q_1}^{\tau^{(1)}\cdots\tau^{(N)}}(\sigma^{(1)},..., \sigma^{(N)})\, T(\sigma^{(1)},..., \sigma^{(N)};\tilde{\sigma}^{(1)},..., \tilde{\sigma}^{(N)}) \quad (4.13)\end{aligned}$$

where

$$T(\sigma^{(1)},..., \sigma^{(N)}; \tilde{\sigma}^{(1)},..., \tilde{\sigma}^{(N)}) = \nu_{Q_1\setminus\mathscr{L}}^{\sigma^{(1)}\cdots\sigma^{(N)};\, \tau^{(1)}\cdots\tau^{(N)}}(\tilde{\sigma}^{(1)}_{Q_1\setminus\mathscr{L}},..., \tilde{\sigma}^{(N)}_{Q_1\setminus\mathscr{L}}) \quad \text{if} \quad \tilde{\sigma}^{(i)}_{\mathscr{L}} = \sigma^{(i)}_{\mathscr{L}} \tag{4.14}$$

$$T(\sigma^{(1)},..., \sigma^{(N)}; \tilde{\sigma}^{(1)},..., \tilde{\sigma}^{(N)}) = 0 \qquad \text{otherwise}$$

It is easy to see, using the DLR equations, that if the event A depends only on the kth configuration $\tilde{\sigma}^{(k)}$, then

$$\tilde{\nu}_{Q_1}^{\tau^{(1)}\cdots\tau^{(N)}}(A) = \mu_{Q_1}^{\delta+,\delta+,\tau^{(k)},\delta+}(A) \tag{4.15}$$

and, moreover, that if the event $A \subset (\Omega_{Q_1})^N$ depends *only* on the values of the spins in $Q_1\setminus\mathscr{L}$, then

$$\nu_{Q_1}^{\tau^{(1)}\cdots\tau^{(N)}}(A) = \sum_{\sigma^{(1)}\cdots\sigma^{(N)}} \nu_{Q_1}(\sigma^{(1)},..., \sigma^{(N)})\, \nu_{Q_1\setminus\mathscr{L}}^{\sigma^{(1)}\cdots\sigma^{(N)};\, \tau^{(1)}\cdots\tau^{(N)}}(A) \tag{4.16}$$

Finally, it is immediate to check, using the monotonicity (1.23) of the measures $\nu_{Q_1}^{\tau^{(1)}\cdots\tau^{(N)}}$ and $\nu_{Q_1\setminus\mathscr{L}}^{\sigma^{(1)}\cdots\sigma^{(N)};\, \tau^{(1)}\cdots\tau^{(N)}}$, that (1.23) holds true also for $\tilde{\nu}_{Q_1}^{\tau^{(1)}\cdots\tau^{(N)}}$. This fact implies, in particular, that if we use the coupling (1.24), (1.25) with the measures $\tilde{\nu}_{Q_1}, \nu_{Q_2},..., \nu_{Q_n}$, then, under this new coupling, any ordered set of initial configurations will stay ordered at any future time.

Let now $S_N \equiv \{t_1,..., t_N\}$ be a fixed ordered sequence with $t_1 = 0$, let $A_i(x)$, A_i, be the events defined in (3.4), and let $q_i = P(A_i \mid \mathscr{B})$, where

$$\mathscr{B} = \{((-)_{t_1}^{\{Q_i\},\delta+})_{Q_1} \in \mathscr{A}_{Q_1}^{+,+,-,+}\} \tag{4.17}$$

As in Section 3, we have

$$q_{n+1} \leqslant q_n + P(A_{n+1} \cap A_n^c \mid \mathscr{B}) \tag{4.18}$$

Let us estimate the second term in the r.h.s. of (4.18). As in Section 3, we let

$$R_n = \left\{ x \in \bigcup_{j \leqslant n} Q_j;\ x_2 \leqslant (n+1)\frac{l}{2} - \left[\frac{l}{4}\right] \right\}$$

and

$$D = \bigcup_{j \geqslant 2} Q_j$$

We observe that, since the sequence S_N is ordered, if A_n^c has occurred for some $n \geqslant 1$, then necessarily $(-)_{t_1}^{\{Q_i\},\delta+}$ and $(+)_{t_1}^{\{Q_i\},\delta+}$ are both equal on the external boundary of the bottom side of Q_2 to a common configuration that we call τ. Again because of the ordering of the sequence S_N, the next updatings at time t_i, $i \geqslant 2$, will not modify τ on the external boundary of the bottom side of Q_2 and therefore they will be reversible with respect to the Gibbs measure $\mu_D^{\tau,\delta+,\delta+,\delta+}$ on Ω_D.

Thus, following Section 3 [see (3.6)–(3.9)], we can bound $P(A_{n+1} \cap A_n^c \mid \mathscr{B})$ by

$$\sum_\tau \tilde{\nu}_{Q_1}^{\tau^{(1)}\cdots\tau^{(N)}}((-)_{t_1}^{\{Q_i\},\delta+} = (+)_{t_1}^{\{Q_i\},\delta+} = \tau \mid \mathscr{A}_{Q_1}^{+,+,-,+})\, F_1(\tau)$$

$$\leqslant \sum_\tau \mu_{Q_1}^{\delta+,\delta+,-,\delta+}(\tau \mid \mathscr{A}_{Q_1}^{+,+,-,+})\, F_1(\tau)$$

where

$$F_1(\tau) \equiv \sum_{\substack{x \in R_{n+1} \cap Q_{n+1} \\ \sigma \in \Omega_D}} \mu_D^{\tau,\delta+,\delta+,\delta+}(\sigma)$$

$$\times [\mu_{Q_{n+1}}^{\sigma,\delta+,+,\delta+}(\eta(x)=1) - \mu_{Q_{n+1}}^{\sigma,\delta+,-,\delta+}(\eta(x)=1)] \quad (4.19)$$

As in (3.9)–(3.11), we get, by monotonicity and the DLR equations, that $F_1(\tau)$ is bounded from above by

$$F_2(\tau) = \sum_{x \in R_{n+1} \cap Q_{n+1}} [\mu_{R_{n+1} \cup Q_{n+1} \setminus Q_1}^{\tau,\delta+,+,\delta+}(\sigma)(\eta(x)=1)$$

$$- \mu_{R_{n+1} \cup Q_{n+1} \setminus Q_1}^{\tau,\delta+,-,\delta+}(\sigma)\,(\eta(x)=1)] \quad (4.20)$$

In conclusion we have shown that

$$P(A_{n+1} \cap A_n^c \mid \mathscr{B}) \leqslant \sum_\tau \mu_{Q_1}^{\delta+,\delta+,-,\delta+}(\tau \mid \mathscr{A}_{Q_1}^{+,+,-,+})\, F_2(\tau) \quad (4.21)$$

In order to prove that (4.21) is very small, we need a last result on the Ising model which shows that, conditional to the event $\mathscr{A}_{Q_1}^{+,+,-,+}$, the projection (or relativization) of the measure $\mu_{Q_1}^{\delta+,\delta+,-,\delta+}$ on the external boundary of the bottom side of Q_2 is, in some sense, very close to the same projection both of the measure $\mu_{R_{n+1} \cup Q_{n+1}}^{\delta+,\delta+,+,\delta+}$ and of the measure $\mu_{R_{n+1} \cup Q_{n+1}}^{+,\delta+,-,\delta+}$. More precisely:

Proposition 4.3. Let $m > 0$ and $\varepsilon \in (0, 1/2)$ be given. Then there exists $\beta_0 \equiv \beta_0(\varepsilon, m)$ such that for all $\beta \geqslant \beta_0$ we have: (a)

$$\left| \sum_\tau \mu_{Q_1}^{\delta+,\delta+,-,\delta+}(\tau \mid \mathscr{A}_{Q_1}^{+,+,-,+})\, F_2(\tau) - \sum_\tau \mu_{R_{n+1} \cup Q_{n+1}}^{\delta+,\delta+,+,\delta+}(\tau)\, F_2(\tau) \right|$$

$$\leqslant \exp(-mL^{1/2+\varepsilon})$$

and (b)

$$\left|\sum_{\tau} \mu_{Q_1}^{\delta+,\delta+,-,\delta+}(\tau \mid \mathscr{A}_{Q_1}^{+,+,-,+}) F_2(\tau) - \sum_{\tau} \mu_{R_{n+1}\cup Q_{n+1}}^{+,\delta+,-,\delta+}(\tau) F_2(\tau)\right|$$
$$\leqslant \exp(-mL^{\varepsilon})$$

For a proof see appendix.

Using Proposition 4.3 and the DLR equation for $\mu_{R_{n+1}\cup Q_{n+1}}^{+,\delta+,-,\delta+}$ and $\mu_{R_{n+1}\cup Q_{n+1}}^{\delta+,\delta+,+,\delta+}$, we get that (4.21) is bounded from above by

$$2\exp(-mL^{\varepsilon}) + \sum_{x\in R_{n+1}\cap Q_{n+1}} [\mu_{R_{n+1}\cup Q_{n+1}}^{\delta+,\delta+,+,\delta+}(\eta(x)=1)$$
$$-\mu_{R_{n+1}\cup Q_{n+1}}^{+,\delta+,-,\delta+}(\eta(x)=1)] \tag{4.22}$$

In turn, using the fact that

$$\mu_{R_{n+1}\cup Q_{n+1}}^{\delta+,\delta+,+,\delta+}(\eta(x)=1) \leqslant \mu_{R_{n+1}\cup Q_{n+1}}^{+,\delta+,+,\delta+}(\eta(x)=1)$$

and applying Proposition 4.1 to the rectangle $R_{n+1}\cup Q_{n+1}$, we get that (4.22) can be bounded from above by

$$3\exp(-mL^{\varepsilon}) \tag{4.23}$$

for any given $m>0$ and $\varepsilon\in(0, 1/2)$, provided that β is large enough depending on m and ε.

In conclusion we have shown that

$$P(A_{n+1}\cap A_n^c \mid \mathscr{B}) \leqslant 3\exp(-mL^{\varepsilon}) \tag{4.24}$$

In order to conclude that q_N is small, we need to control the first term q_1 since

$$q_N \leqslant q_1 + \sum_{n}^{N-1} P(A_{n+1}\cap A_n^c \mid \mathscr{B})$$

Proposition 4.4. Let $m>0$ and $\varepsilon\in(0, 1/2)$ be given. Then there exists $\beta_0\equiv\beta_0(\varepsilon, m)$ such that for all $\beta\geqslant\beta_0$ we have

$$q_1 \leqslant \exp(-mL^{1/2+\varepsilon})$$

Proof. Let $\nu\equiv\tilde{\nu}_{Q_1}$ be the measure on $\Omega_{Q_1}^{2L}$ constructed in (4.13). By monotonicity in the initial configuration and by the definition of the event A_1, we can estimate q_1 from above by

$$q_1 \leqslant \sum_{x\in R_1} [\nu(\sigma^{(N)}(x)=1 \mid \{\sigma^{(1)}\in\mathscr{A}_{Q_1}^{+,+,-,+}\})$$
$$-\nu(\sigma^{(1)}(x)=1 \mid \{\sigma^{(1)}\in\mathscr{A}_{Q_1}^{+,+,-,+}\})] \tag{4.25}$$

where we used the convention that $\sigma^{(1)}$ and $\sigma^{(N)}$ are the components of a generic configuration $\tilde{\sigma} \in \Omega_{Q_1}^{2L}$ corresponding, respectively, to the minimal $(-)$ and maximal $(+)$ boundry conditions on the top side of Q_1.

Let us examine separately each one of the two terms appearing in the sum in the r.h.s. of (4.25).

Because of (4.15), the second term $\nu(\sigma^{(1)}(x)=1 \mid \{\sigma^{(1)} \in \mathscr{A}_{Q_1}\})$ is equal to

$$\begin{aligned}\nu(\sigma^{(1)}(x)=1 \mid \{\sigma^{(1)} \in \mathscr{A}_{Q_1}^{+,+,-,+}\}) \\ = \mu_{Q_1}^{\delta+,\delta+,-,\delta+}(\sigma(x)=1 \mid \mathscr{A}_{Q_1}^{+,+,-,+})\end{aligned} \tag{4.26}$$

Since the event $\mathscr{A}_{Q_1}^{+,+,-,+}$ implies that the entire unique open contour of the configuration σ is *outside* R_1, it is immediate to check, using the monotonicity of the Gibbs measure with respect to an increase of the boundary conditions, that

$$\mu_{Q_1}^{\delta+,\delta+,-,\delta+}(\sigma(x)=1 \mid \mathscr{A}_{Q_1}^{+,+,-,+}) \geqslant \mu_{Q_1}^{\delta+,\delta+,+,\delta+}(\sigma(x)=1) \tag{4.27}$$

Let us now consider the first term

$$\begin{aligned}\nu(\sigma^{(N)}(x)=1 \mid \{\sigma^{(1)} \in \mathscr{A}_{Q_1}^{+,+,-,+}\}) \\ = \frac{\nu(\sigma^{(N)}(x)=1 \cap \{\sigma^{(1)} \in \mathscr{A}_{Q_1}^{+,+,-,+}\})}{\mu_{Q_1}^{\delta+,\delta+,-,\delta+}(\mathscr{A}_{Q_1}^{+,+,-,+})}\end{aligned} \tag{4.28}$$

where we used, once more, (4.15).

We observe that the event $\{\sigma(x)=1\}$, $x \in R_1$, and $\mathscr{A}_{Q_1}^{+,+,-,+}$ depend *only* on the spins in $R_1 \subset \tilde{R}_1$ and $Q_1 \backslash \{\tilde{R}_1 \cup \mathscr{L}\}$, respectively, where $\tilde{R}_1$ and $\mathscr{L}$ have been defined right after Proposition 4.2.

Therefore, using (4.16) and the fact that $\nu_{Q_1 \backslash \mathscr{L}}^{\sigma^{(1)} \dots \sigma^{(N)};\, \tau^{(1)} \dots \tau^{(N)}}$ is a product of the measures $\nu_{\tilde{R}_1}^{\sigma^{(1)} \dots \sigma^{(N)}}$ and $\nu_{Q_1 \backslash \{\tilde{R}_1 \cup \mathscr{L}\}}^{\sigma^{(1)} \dots \sigma^{(N)};\, \tau^{(1)} \dots \tau^{(N)}}$, we get

$$\begin{aligned}\nu(\sigma^{(N)}(x)=1 \cap \{\sigma^{(1)} \in \mathscr{A}_{Q_1}^{+,+,-,+}\}) \\ = \sum_{\sigma^{(1)} \dots \sigma^{(N)}} \nu_{Q_1}(\sigma^{(1)},\dots,\sigma^{(N)})\, \nu_{\tilde{R}_1}^{\sigma^{(1)} \dots \sigma^{(N)}}(\tilde{\sigma}^{(N)}(x)=1) \\ \times \nu_{Q_1 \backslash \{\tilde{R}_1 \cup \mathscr{L}\}}^{\sigma(1) \dots \sigma^{(N)};\, \tau^{(1)} \dots \tau^{(N)}}(\tilde{\sigma}^{(1)} \in \mathscr{A}_{Q_1}^{+,+,-,+})\end{aligned} \tag{4.29}$$

We now observe that, because of (1.22) applied to $\nu_{\tilde{R}_1}^{\sigma^{(1)} \dots \sigma^{(N)}}$,

$$\begin{aligned}\nu_{\tilde{R}_1}^{\sigma^{(1)} \dots \sigma^{(N)}}(\tilde{\sigma}^{(N)}(x)=1) = \mu_{\tilde{R}_1}^{\delta+,\delta+,\sigma^{(N)},\delta+}(\sigma(x)=1) \\ \leqslant \mu_{\tilde{R}_1}^{\delta+,\delta+,+,\delta+}(\sigma(x)=1)\end{aligned} \tag{4.30}$$

so that the r.h.s. of (4.29) becomes smaller than

$$\mu_{\tilde{R}_1}^{\delta+,\delta+,+,\delta+}(\sigma(x)=1)\, \nu(\sigma^{(1)}\in \mathscr{A}_{Q_1}^{+,+,-,+})$$
$$= \mu_{\tilde{R}_1}^{\delta+,\delta+,+,\delta+}(\sigma(x)=1)\, \mu_{Q_1}^{\delta+,\delta+,-,\delta+}(\sigma\in \mathscr{A}_{Q_1}^{+,+,-,+}) \tag{4.31}$$

where we used once more (4.15) to write

$$\nu(\sigma^{(1)}\in \mathscr{A}_{Q_1}^{+,+,-,+}) = \mu_{Q_1}^{\delta+,\delta+,-,\delta+}(\sigma\in \mathscr{A}_{Q_1}^{+,+,-,+})$$

In conclusion, from (4.28)–(4.31), we get that

$$\nu(\sigma^{(N)}(x)=1 \mid \{\sigma^{(1)}\in \mathscr{A}_{Q_1}^{+,+,-,+}\}) \leqslant \mu_{\tilde{R}_1}^{\delta+,\delta+,+,\delta+}(\sigma(x)=1) \tag{4.32}$$

Combining finally (4.27) and (4.32), we bound from above the sum in (4.25) by

$$\sum_{x\in R_1} [\mu_{\tilde{R}_1}^{\delta+,\delta+,+,\delta+}(\sigma(x)=1) - \mu_{Q_1}^{\delta+,\delta+,+,\delta+}(\sigma(x)=1)]$$
$$\leqslant \exp(-mL^{1/2+\varepsilon}) \tag{4.33}$$

for any given m, provided that β is large enough. In the derivation of the last inequality in (4.33) we use part (ii) of Proposition 4.1 and the definition of $\tilde{R}_1$.

If we now use Proposition 4.4 together with (4.24), we get that

$$P(A_N \mid \mathscr{B}) \leqslant 3N \exp(-mL^{\varepsilon}) \tag{4.34}$$

We are now in a position to conclude the proof of the theorem. Given a sequence $S_N \equiv \{t_1, ..., t_N\}$ of updatings, we say that S_N is a *good* sequence iff S_N is ordered and the event A_N^c occurred at the end of the sequence. Using (4.34) together with Proposition 4.2, we conclude that the probability that an ordered sequence is also a good sequence is larger than

$$[1 - N\exp(-mL^{\varepsilon})]\, P(\mathscr{B}) \geqslant \tfrac{1}{2}\exp[-\beta L\tau_\beta - C\beta L^{1/2+\varepsilon})]$$

for L large enough and some constant C.

Thus, using Lemma 3.1, we get that, if $T = \exp[+\beta L\tau_\beta + 2C\beta L^{1/2+\varepsilon})]$ and L is large enough

$$P(\text{there exists a good sequence in } [0, T]) \geqslant \tfrac{1}{3} \tag{4.35}$$

As in Section 3, (4.35) immediately implies that

$$\text{gap}_{\nu_L}(\{Q_i\}, \varnothing) \geqslant \exp[-\beta L\tau_\beta - 3C\beta L^{1/2+\varepsilon})] \tag{4.36}$$

Clearly (4.36) together with (4.10) and (4.11) proves the correct lower bound.

The proof of the theorem is completed.

5. RARE EXCURSIONS OF THE MAGNETIZATION

In this section we apply the results obtained in the previous sections to study in detail the time evolution of the magnetization $m(\sigma_t)$ of the process. In particular we will analyze the large fluctuations of the observable $m(\sigma_t)$ and prove some asymptotic results close in spirit to the results obtained by Shlosman for the static problem (see Theorem 4.2).

The setting will be that of Section 4, namely the HB-dynamics in a square $V \equiv V_L$ of side L with open boundary conditions. Although the case with plus boundary conditions could be treated as well without any significant modification, we decided to omit it in order not to burden too much the reader.

Let ρ_L, $L \in \mathcal{N}$, be a sequence of integers such that

$$\lim_{L \to \infty} \frac{\rho_L}{L^2} = \rho \in (-m^*(\beta), m^*(\beta)), \qquad \rho_L - L^2 = 0 \mod 2$$

where, as usual, $m^*(\beta)$ denotes the spontaneous magnetization, and let τ_{ρ_L} be the stopping time:

$$\tau_{\rho L} = \inf\{t \geqslant 0;\ m(\sigma_t) \leqslant \rho_L\} \tag{5.1}$$

Then our two main results can be stated as follows:

Theorem 5.1. There exists β_0 such that for any $\beta \geqslant \beta_0$ and any ρ_L as above

$$\lim_{L \to \infty} \frac{1}{\beta L} \log\left[\sum_{\substack{\sigma \\ m(\sigma) > 0}} \mu_V^{\varnothing}(\sigma)\, E_\sigma(\tau_{\rho_L}) \right] = \psi(\rho \vee 0)$$

where the symbol E_σ denotes the expectation over the HB-dynamics starting from the configuration σ and the function $\psi(\rho)$ is given in Theorem 4.2.

The same asymptotics holds if instead of starting from equilibrium with positive magnetization we start from the configuration identically equal to all pluses.

Theorem 5.2. There exists β_0 such that for any $\beta \geqslant \beta_0$ and any ρ_L as above, there exist numbers $\{a_L\}_{L \in \mathbb{N}}$ such that for any $t > 0$: (a)

$$\lim_{L \to \infty} \frac{1}{\beta L} \log(a_L) = \psi(\rho \vee 0)$$

and (b)

$$\lim_{L \to \infty} \sum_{\substack{\sigma \\ m(\sigma) > 0}} \mu_V^{\varnothing}(\sigma)\, P_\sigma(\tau_{\rho_L} > t a_L) = \exp(-t)$$

An analogous result holds if instead of starting from equilibrium we start from the configuration identically equal to all pluses.

Remark. Theorem 5.2 says that, under a suitable rescaling determined by the numbers $a_L \approx \exp[\beta L \psi(\rho)]$, the stopping time τ_{ρ_L} started at equilibrium with positive magnetization becomes essentially *unpredictable*, i.e., it can be thought of as the (random) number of independent attempts, each of which has a probability of success of the order of $\exp[-\beta L(\psi)(\rho)]$, that one has to make before seeing a success.

For results with the magnetization density ρ outside the region $(-m^*(\beta), +m^*(\beta))$ we refer the reader to the paper by Lebowitz and Schonmann.[15]

Proof of Theorem 5.1. We start by proving a lower bound of the right order when we start from the measure $\mu_V^{\varnothing}$ restricted to the configurations of positive magnetization.

Clearly for such class of configurations

$$\tau_{\rho_L} \geqslant \tau_{\rho_L \vee 0}$$

so that it is enough to prove a correct lower bound only for $\rho \in [0, m^*(\beta))$.

For any positive T we can write

$$\begin{aligned}\sum_{\substack{\sigma \\ m(\sigma) > 0}} \mu_V^{\varnothing}(\sigma)\, E_\sigma(\tau_{\rho_L}) &\geqslant T \sum_{\substack{\sigma \\ m(\sigma) > 0}} \mu_V^{\varnothing}(\sigma)\, [1 - P_\sigma(\tau_{\rho_L} < T)] \\ &\geqslant \frac{T}{2} - T \sum_{\substack{\sigma \\ m(\sigma) > 0}} \mu_V^{\varnothing}(\sigma)\, P_\sigma(\tau_{\rho_L} \leqslant T) \end{aligned} \qquad (5.2)$$

where we used the symmetry of the Gibbs measure under global spin flip.

As in Section 4 [see (4.4) and (4.5)] the sum in the r.h.s. of (5.2) can be estimated from above by

$$2L^2 T \mu_V^{\varnothing}(m(\sigma) = \rho_L) + \exp(-KL^2 T) \qquad (5.3)$$

for a suitable constant K.

We now take the time T of the form

$$T = \exp[\beta L(\psi(\rho) - \delta)] \qquad (5.4)$$

where δ is any fixed small number independent of L. If we recall Theorem 4.2, we get that, with this choice of T, (5.3) goes to zero as L gets large. This fact together with (5.2) implies that

$$\lim_{L\to\infty} \frac{1}{\beta L} \log\left[\sum_{\substack{\sigma \\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\, E_\sigma(\tau_{\rho_L})\right] \geqslant \psi(\rho)-\delta \quad \forall \rho\in[0, m_\beta^*) \tag{5.5}$$

Since δ can be taken arbitrarily small (after the limit $L\to\infty$), the required lower bound follows. It is also clear that the same lower bound applies also to $E_+(\tau_{\rho_L})$, that is, when the starting configuration is identically equal to plus one, since, because of monotonicity in the initial configuration,

$$E_+(\tau_{\rho_L}) \geqslant \sum_{\substack{\sigma \\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\, E_\sigma(\tau_{\rho_L})$$

In order to prove an upper bound, we have to distinguish between two cases:

Case 1 $\quad \rho\in(-m^*(\beta), \rho_1]$

Case 2 $\quad \rho\in(\rho_1, m^*(\beta))$

where ρ_1 is the singularity point of the function $\psi(\rho)$ defined in Theorem 4.2.

Let us begin with the first one.

Clearly

$$\begin{aligned}\sum_{\substack{\sigma \\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\, E_\sigma(\tau_{\rho_L}) &= \sum_{\substack{\sigma \\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma) \sum_n P_\sigma(\tau_{\rho_L}\geqslant n)\\ &\leqslant \sum_n P_+(\tau_{\rho_L}\geqslant n)\end{aligned} \tag{5.6}$$

since, by monotonicity in the initial configuration,

$$P_\sigma(\tau_{\rho_L}\geqslant n) \leqslant P_+(\tau_L\geqslant n) \quad \forall n \quad \forall \sigma \tag{5.7}$$

In turn, for any integer N, it follows from the Markov property and (5.7) that

$$P_+(\tau_{\rho_L}\geqslant n) \leqslant P_+(\tau_{\rho_L}\geqslant N)^{[n/N]} \tag{5.8}$$

Let us therefore estimate $P_+(\tau_{\rho_L}\geqslant N)$. We write

$$\begin{aligned}P_+(\tau_{\rho_L}\geqslant N) &= P_+\left(\int_0^N dt_\chi(m(\sigma_t)\geqslant \rho_L) = N\right)\\ &\leqslant \frac{\int_0^N dt\, P_+(m(\sigma_t)\geqslant \rho_L)}{N}\end{aligned} \tag{5.9}$$

where $\chi(A)$ denotes the characteristic function of the event A and we used the generalized Chebyshev inequality in order to get the last inequality.

Let us now choose the integers N, N_0 equal to

$$N = \exp[\beta L(\psi(0) + 2\delta)]$$
$$N_0 = \exp[\beta L(\psi(0) + \delta)], \qquad \delta \ll 1$$

Then we can estimate the integral in the r.h.s. of (5.9) by

$$\begin{aligned}\frac{\int_0^N dt\, P_+(m(\sigma_t) \geqslant \rho_L)}{N} &\leqslant \frac{N_0}{N} + \mu_V^{\varnothing}(m(\sigma) \geqslant \rho_L) \\ &+ \frac{\int_{N_0}^N dt [P_+(m(\sigma_t) \geqslant \rho_L) - \mu_V^{\varnothing}(m(\sigma) \geqslant \rho_L)]}{N}\end{aligned} \tag{5.10}$$

Let us examine separately each one of the three terms in the r.h.s. of (5.10) in the limit as $L \to \infty$. The first term goes to zero by construction. The second term converges to 1/2 because of Theorem 4.2. The third term also goes to zero for β large enough, if we use Theorem 4.1, the basic estimate (1.19), the fact that $\psi(0) = \tau_\beta$, and our choice of the integer N_0.

In conclusion we have shown that, for all β large enough and all large enough L,

$$P_+(\tau_{\rho_L} \geqslant N) \leqslant \tfrac{2}{3} \tag{5.11}$$

Clearly (5.11) together with (5.6), (5.8) proves that

$$\begin{aligned}\sum_{\substack{\sigma \\ m(\sigma) > 0}} \mu_V^{\varnothing}(\sigma)\, E_\sigma(\tau_{\rho_L}) \leqslant E_+(\tau_{\rho_L}) \leqslant 3N \\ = 3\exp[\beta L(\psi(0) + 2\delta)]\end{aligned} \tag{5.12}$$

which establishes the correct upper bound in the limit $L \to \infty$ due to the arbitrariness of δ also for the case when the starting configuration is identically equal to plus one.

Let us now treat the (more difficult) second case $\rho \in [\rho_1, m^*(\beta))$.

First of all we bound

$$\sum_{\substack{\sigma \\ m(\sigma) > 0}} \mu_V^{\varnothing}(\sigma)\, E_\sigma(\tau_{\rho_L})$$

by the same quantity but computed for the HB-dynamics in V with extra plus boundary conditions on the top horizontal side and starting from all pluses:

$$\sum_{\substack{\sigma \\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\, E_\sigma(\tau_{\rho_L}) \leqslant E_+^{\varnothing,\varnothing,+,\varnothing}(\tau_{\rho_L}) \tag{5.13}$$

with self-explanatory notation. The reason for introducing on only one side of V extra plus boundary conditions is the following. For large L, the relaxation time to equilibrium $[\equiv \mathrm{gap}(\mathrm{HB}, \varnothing, \varnothing, +, \varnothing)^{-1}]$ with the indicated boundary conditions is of the order of $\exp(C\beta L^{1/2+\varepsilon})$ (see Corollary 4.1); therefore the relaxation time is much smaller than the inverse of the equilibrium measure of the hitting set $\{\sigma; m(\sigma) \leqslant \rho_L\}$,

$$\mu_V^{\varnothing,\varnothing,+,\varnothing}(m(\sigma)\leqslant \rho_L)^{-1} \geqslant \exp[\beta c(\rho) L]; \qquad c(\rho)>0$$

It thus follows from a standard argument (see, e.g., ref. 1) that

$$E_+^{\varnothing,\varnothing,+,\varnothing}(\tau_{\rho_L}) \leqslant \mu_V^{\varnothing,\varnothing,+,\varnothing}(m(\sigma)\leqslant \rho_L)^{-1} \exp(\beta\, \delta L), \qquad \delta \ll 1 \tag{5.14}$$

Thus one needs, in strict analogy with Theorem 4.2, to estimate from below

$$\mu_V^{\varnothing,\varnothing,+,\varnothing}(m(\sigma)\leqslant \rho_L)$$

as $L\to\infty$. This is the content of the next proposition:

Proposition 5.1. Let ρ_L be as above. Then there exists β_0 such that for any $\beta \geqslant \beta_0$ and any given positive δ

$$\mu_V^{\varnothing,\varnothing,+,\varnothing}(m(\sigma)\leqslant \rho_L) \geqslant \exp[-\beta(\psi(\rho)+\delta)L]$$

for all L large enough, where $\psi(\rho)$ is as in Theorem 4.2.

The proposition can be proved by exactly the same methods as developed in ref. 4 (see also ref. 22) and employed by Shlosman[26] in his proof of Theorem 4.2; the proof is, however, lengthy and therefore it is not included in this work.

It is possible to give a convincing explanation why the extra plus boundary conditions on the top side of V do not affect the asymptotics (or at least a lower bound) of

$$\mu_V^{\varnothing,\varnothing,+,\varnothing}(m(\sigma)\leqslant \rho_L)$$

In ref. 26 (see ref. 4 for full details in the case of periodic boundary conditions and ref. 22 in the case of plus boundary conditions) the asymptotics of $\mu_V^{\varnothing,\varnothing,\varnothing,\varnothing}(m(\sigma)=\rho_L)$ is derived by proving that the typical structure of the set of configurations under the event $\{m(\sigma)=\rho_L\}$ is as follows:

1. In case $\rho>\rho_1$ there is a bubble W_ρ^0 of the minus phase close to one of the four corners of V, while in $V\backslash W_\rho^0$ one has the plus phase. The shape of W_ρ^0 is that of the intersection with V of the rescaled Wulff shape $2[(m_\beta^*-\rho)/2m_\beta^*]^{1/2}\, W$ of total volume $4(m_\beta^*-\rho)/2m_\beta^*$ and centered at one of the corners of V. It is clear from the results in ref. 4 (see Chapter 5) that the probability (with open boundary conditions) for the above situation to occur is of the order

$$\exp\left[-\beta L\frac{1}{2}\left(\frac{m_\beta^*-\rho}{2m_\beta^*}\right)w\right]=\exp[-\beta\psi(\rho)L]$$

where w is the Wulff functional computed on the Wulff curve ∂W.

2. In case $\rho\leqslant\rho_1$, where ρ_1 is as in Theorem 4.2, it is more convenient to divide the volume V into roughly two rectangles, with the correct volumes determined by ρ, by means of a (roughly) straight horizontal line. It is clear that in this other case the probability is of the order of

$$\exp(-\beta\tau_\beta L)$$

for any $\rho\leqslant\rho_1$.

In the first case, we can impose on our configuration that it has a unique "large" contour exactly like the one described above, at distance greater than cL from the top side of V, where $c>1/2$ is a suitable constant depending on ρ. Since the coefficients $\Phi^{\varnothing,\varnothing,+,\varnothing}(\Lambda)$ of the cluster expansion of the partition function decay exponentially fast in the "size" of the set Λ, it is not difficult to see that in this way one obtains a lower bound on $\mu_V^{\varnothing,\varnothing,+,\varnothing}(m(\sigma)\leqslant\rho_L)$ which, apart from minor corrections that are adsorbed in the δ appearing in the proposition, is like the one obtained without the extra plus boundary condition on the top side.

It is clear that if we plug the statement of the proposition into (5.14) we get the required upper bound. The proof of the theorem is complete.

Proof of Theorem 5.2. Let us define the numbers a_L by the condition

$$\sum_{\substack{\sigma\\ m(\sigma)>0}}\mu_V^{\varnothing}(\sigma)\,P_\sigma(\tau_{\rho_L}\geqslant a_L)=e^{-1}\tag{5.15}$$

and let $f_L(t)$ be given by

$$f_L(t)=\sum_{\substack{\sigma\\ m(\sigma)>0}}\mu_V^{\varnothing}(\sigma)\,P_\sigma(\tau_{\rho_L}>a_L t)\tag{5.16}$$

In order to prove the theorem it is enough, using the normalization (5.15), to show that

$$\lim_{L\to\infty} |f_L(t+s)-f_L(t)\, f_L(s)| = 0 \tag{5.17}$$

and that the asymptotics of the number a_L, as $L\to\infty$, is the right one. Because of (5.2) applied to $T=a_L$, we immediately get that

$$a_L \leqslant e \sum_{\substack{\sigma\\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\, E_\sigma(\tau_{\rho_L}) \tag{5.18}$$

In order to obtain a lower bound on a_L we observe that, using the argument employed in Section 4 [see, e.g., (4.4), (4.5)]

$$\begin{aligned} 1-e^{-1} &= \sum_{\substack{\sigma\\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\, P_\sigma(\tau_{\rho_L}<a_L)\\ &\leqslant 2L^2(a_L \vee 1)\, \mu_V^{\varnothing}(m(\sigma)=(\rho_L \vee 0)) + \exp[-KL^2(a_L\vee 1)] \end{aligned} \tag{5.19}$$

for a suitable constant K. Thus

$$a_L \geqslant \frac{1-e^{-1}}{4L^2\mu_V^{\varnothing}(m(\sigma)=(\rho_L\vee 0))} \tag{5.20}$$

for large L. Clearly (5.18) and (5.20) together with Theorems 5.1 and 4.2 prove the first part of the theorem.

Let us turn to the proof of (5.17). We observe that, because of the definition of the stopping time τ_{ρ_L}, it trivially follows that

$$\begin{aligned} f_L(t) &= \sum_\sigma \mu_V^{\varnothing}(\sigma)\, P_\sigma(\tau_{\rho_L}>a_L t)\\ &\quad - \sum_{\substack{\sigma\\ \rho_L\leqslant m(\sigma)\leqslant 0}} \mu_V^{\varnothing}(\sigma)\, P_\sigma(\tau_{\rho_L}>a_L t) \equiv \bar f_L(t) - \varepsilon_L \end{aligned}$$

Clearly, using Theorem 4.2, ε_L goes to zero exponentially fast in L.

Using the reversibility of the dynamics with respect to the Gibbs measure $\mu_V^{\varnothing}$, we can write $\bar f_L(t+s)$ as

$$\begin{aligned} \bar f_L(t+s) &= \sum_\sigma \mu_V^{\varnothing}(\sigma)\, P_\sigma(\tau_{\rho_L}>a_L t)\, P_\sigma(\tau_{\rho_L}>a_L s)\\ &= \sum_{\substack{\sigma\\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\, P_\sigma(\tau_{\rho_L}>a_L t)\, P_\sigma(\tau_{\rho_L}>a_L s) + \varepsilon_L \end{aligned}$$

so that the difference $|f_L(t+s)-f_L(t)\,f_L(s)|$ can be estimated from above by

$$\Bigg|\sum_{\substack{\sigma,\eta\\ m(\sigma)>0,\, m(\eta)>0}} \mu_V^{\varnothing}(\sigma)\,\mu_V^{\varnothing}(\eta)\,P_\sigma(\tau_{\rho_L}>a_L t) \times [P_\sigma(\tau_{\rho_L}>a_L s)-P_\eta(\tau_{\rho_L}>a_{L_s})]\Bigg|+2\varepsilon_L \tag{5.21}$$

If we now couple the HB-dynamics starting from σ and η together in the way described in Section 1, we can estimate the first term in (5.21) by

$$\sum_{\substack{\sigma,\eta\\ m(\sigma)>0,\, m(\eta)>0}} \mu_V^{\varnothing}(\sigma)\,\mu_V^{\varnothing}(\eta)\,P(\tau_{\rho_L}(\sigma)\neq\tau_{\rho_L}(\eta)) \tag{5.22}$$

where, with an abuse of notation, P denotes the probability measure of the coupled process, and $\tau_{\rho_L}(\sigma)$ and $\tau_{\rho_L}(\eta)$ the stopping times starting from σ and η, respectively.

The idea behind the estimate of $P(\tau_{\rho_L}(\sigma)\neq\tau_{\rho_L}(\eta))$ (see below) is at this point very natural: when the two starting configurations σ and η are both chosen at random with respect to the Gibbs measure $\mu_V^{\varnothing}$ restricted to the "phase" $\{m>0\}$, then in a time scale T_0, which is much shorter than the typical time scale of $\tau_{\rho_L}(\sigma)$ and $\tau_{\rho_L}(\eta)$, the two configurations become identical with very large probability. The reason for this quick loss of memory inside the "phase" $\{m>0\}$, in contrast to the smallness of the gap (see Theorem 4.1), has to be found in the fact (see the proof of Proposition 5.2 below) that, starting in equilibrium with positive magnetization, with large probability the HB-dynamics in V with open boundary conditions cannot be distinguished, at a given site $x\in V$, from the HB-dynamics in V with an extra plus boundary condition on one of the sides of V. This latter loses memory of the initial condition much faster than the dynamics with open boundary conditions (see Corollary 4.1) and the result follows.

Let us start with the technicalities. Let $\varepsilon\in(0,1/2)$ be given and let $T_0=\exp(\beta L^{1/2+\varepsilon})$. Then we estimate (5.22) by

$$2\sum_{\substack{\sigma\\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\,P_\sigma(\tau_{\rho_L}<T_0)+2\sum_{\substack{\sigma\\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\,P(\sigma_{T_0}\neq(+)_{T_0}) \tag{5.23}$$

where $(+)_{T_0}$ is the evolute at time T_0 of the configuration identically equal to $+1$.

We know already [see (5.19)] that the first term in (5.23) goes to zero as $L\to\infty$ provided that β is large enough. The second term is controled by the following new result:

Proposition 5.2. Let $\varepsilon \in (0, 1/2)$ and $m > 0$ be given. Then there exist $\beta_0 < +\infty$ and $C < +\infty$ such that for any $\beta \geqslant \beta_0$, any integer L, and any time $t \geqslant \exp(C\beta L^{1/2+\varepsilon})$

$$\sum_{\substack{\sigma \\ m(\sigma) > 0}} \mu_V^{\varnothing}(\sigma)\, P((+)_t \neq \sigma_t) \leqslant \exp(-mL)$$

It is clear that the above proposition concludes the proof of Theorem 5.2 in the case the starting configuration is distributed according to the restriction of the Gibbs measure to the set $\{\sigma; m(\sigma) > 0\}$. A similar argument can be repeated if the starting configuration is identically equal to $+1$.

Proof of Proposition 5.2. Since

$$P((+)_{t+s} \neq \sigma_{t+s}) \leqslant P((+)_t \neq \sigma_t) \quad \forall s \geqslant 0$$

it is sufficient to prove the result for the fixed time $t_o = \exp(C\beta L^{1/2+\varepsilon})$. We first estimate $P((+)_{t_0} \neq \sigma_{t_0})$ by

$$P((+)_{t_0} \neq \sigma_{t_0}) \leqslant \sum_{x \in V} P((+)_{t_0}(x) \neq \sigma_{t_0}(x))$$

Given now $x \in V$, let us uppose, without loss of generality, that the top horizontal side of V is such that its distance from x is greater than or equal to $L/2$. Let also $(+)_{t_0}^{\varnothing, \varnothing, +, \varnothing}$ be the evoluted at time t_0 of the configuration $(+)$ under the HB-dynamics in V with $(\varnothing, \varnothing, +, \varnothing)$ boundary conditions on $\partial_{\text{ext}} V$. Then, by monotonicity, we have

$$\begin{aligned} &P((+)_{t_0}(x) \neq \sigma_{t_0}(x)) \\ &\quad \leqslant P((+)_{t_0}^{\varnothing, \varnothing, +, \varnothing}(x) \neq \sigma_{t_0}(x)) \\ &\quad = P((+)_{t_0}^{\varnothing, \varnothing, +, \varnothing}(x) = +1) - P(\sigma_{t_0}(x) = +1) \end{aligned} \tag{5.24}$$

Thus

$$\begin{aligned} &\sum_{\substack{\sigma \\ m(\sigma) > 0}} \mu_V^{\varnothing}(\sigma)\, P((+)_{t_0} \neq \sigma_{t_0}) \\ &\quad \leqslant \sum_{x \in V} \left\{ \sum_{\substack{\sigma \\ m(\sigma) > 0}} \mu_V^{\varnothing}(\sigma) \left[P((+)_{t_0}^{\varnothing, \varnothing, +, \varnothing}(x) = +1) - P(\sigma_{t_0}(x) = +1)\right] \right\} \\ &\quad = \sum_{x \in V} \left[\frac{1}{2} P((+)_{t_0}^{\varnothing, \varnothing, +, \varnothing}(x) = +1) \right. \\ &\qquad \left. - \sum_{\substack{\sigma \\ m(\sigma) > 0}} \mu_V^{\varnothing}(\sigma)\, P(\sigma_{t_0}(x) = +1) \right] \end{aligned} \tag{5.25}$$

Let us first treat the term $P((+)_{t_0}^{\varnothing,\varnothing,+,\varnothing}(x)=+1)$. Using (1.19), Corollary 4.1, and our choice of the time t_0, we get that

$$0 \leqslant P((+)_{t_0}^{\varnothing,\varnothing,+,\varnothing}(x)=+1)-\mu_V^{\varnothing,\varnothing,+,\varnothing}(\sigma(x)=+1)$$
$$\leqslant \exp[C'L^2 - t_0 \exp(-C\beta L^{(1+\varepsilon)/2})] \leqslant \tfrac{1}{3}\exp(-mL) \tag{5.26}$$

for any given $m>0$ and any $L\in \mathbf{N}$, provided that β is large enough.

As far as the second term in the square brackets in the r.h.s. of (5.25) is concerned, we write

$$\sum_{\substack{\sigma \\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\, P(\sigma_{t_0}(x)=+1)$$
$$\geqslant \sum_{\substack{\sigma \\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\, P(\sigma_{t_0}(x)=+1 \cap m(\sigma_{t_0})>0)$$
$$=\sum_{\sigma} \mu_V^{\varnothing}(\sigma)\, P(\sigma_{t_0}(x)=+1 \cap m(\sigma_{t_0})>0)$$
$$-\sum_{\sigma;\, m(\sigma)\leqslant 0} \mu_V^{\varnothing}(\sigma)\, P(\sigma_{t_0}(x)=+1 \cap m(\sigma_{t_0})>0)$$
$$=\mu_V^{\varnothing}(\sigma(x)=1 \cap m(\sigma)\geqslant 0)$$
$$-\sum_{\sigma;\, m(\sigma)<0} \mu_V^{\varnothing}(\sigma)\, P(\sigma_{t_0}(x)=+1 \cap m(\sigma_{t_0})\geqslant 0) \tag{5.27}$$

where we used the invariance of the measure $\mu_V^{\varnothing}$.

The last term in the r.h.s. of (5.27) can be bounded from above by

$$\sum_{\sigma} \mu_V^{\varnothing}(\sigma)\, P(\text{there exists } s\leqslant t_0;\ m(\sigma_s)=0)$$
$$\leqslant 2L^2 t_0 \mu_V^{\varnothing}(m(\sigma)=0)+\exp(-KL^2 t_0) \tag{5.28}$$

for a suitable constant K and large enough β, by the argument illustrated in Section 4 [see (4.5)].

Clearly, because of our choice of t_0 and of Theorem 4.2, the r.h.s. of (5.28) is smaller than $\frac{1}{3}\exp(-mL)$ for any given m, provided β is large enough.

In conclusion we have shown that

$$\tfrac{1}{2} P\left((+)_{t_0}^{\varnothing,\varnothing,+,\varnothing}(x) - \sum_{\substack{\sigma \\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\, P(\sigma_{t_0}(x)=+1)\right)$$
$$\leqslant \tfrac{1}{2}|\mu_V^{\varnothing,\varnothing,+,\varnothing}(\sigma(x)=+1)-\mu_V^{\varnothing}(\sigma(x)=1\,|\,m(\sigma)\geqslant 0)|$$
$$+\tfrac{2}{3}\exp(-mL) \tag{5.29}$$

for any L, provided that β is large enough.

In order to complete the proof, we need a last, rather obvious result on the Ising model, whose proof is an exercise in the cluster expansion and it is therefore omitted.

Lemma 5.1. Given $m>0$, there exists β_0 such that for all $\beta>\beta_0$ and all L

$$|\mu_V^{\varnothing,\varnothing,+,\varnothing}(\sigma(x)=+1)-\mu_V^{\varnothing}(\sigma(x)=1\,|\,m(\sigma)\geqslant 0)|\leqslant \tfrac{1}{3}\exp(-mL)$$

If we apply the lemma to (5.29), we obtain

$$\sum_{x\in V}\sum_{\substack{\sigma\\ m(\sigma)>0}} \mu_V^{\varnothing}(\sigma)\,P((+)_t\neq\sigma_t)\leqslant L^2\exp(-mL)$$

for any given $m>0$ and any $L\in\mathbf{N}$, provided that β is large enough. The proposition is proved.

6. MARKOV CHAIN DESCRIPTION OF THE TIME-RESCALED MAGNETIZATION

In this final section we work in the same setting and notation of the previous two sections and we consider the normalized magnetization

$$\rho(\sigma_t)=\frac{1}{|V_L|}\sum_{x\in V_L}\sigma_t(x)$$

of the process started at equilibrium [or from one of the two extreme configurations $(+)$ or $(-)$).

We show that it is possible to rescale the time t by a multiplicative factor t_L depending on the side L of the square V_L, in such a way that, as $L\to\infty$, the finite-dimensional distributions of the rescaled process

$$\rho(\tilde{\sigma}_t)=\rho(\sigma_{t_L t}) \tag{6.1}$$

converge to those of a continuous-time Markov chain on the set $\{-m^*(\beta), m^*(\beta)\}$ with unitary jump rate for both states.

From what we just said, it is clear that the speeding factor t_L must be determined essentially by the condition that

$$\int_{m(\sigma)>0} d\mu_{V_L}^{\varnothing}(\sigma)\,P(\rho(\sigma_t)\approx -m^*(\beta))\approx\frac{p}{2}$$

where

$$p=\tfrac{1}{2}[1-\exp(-2)] \tag{6.2}$$

is the probability that a continuous-time Markov chain with unitary jump rate on $\{-1, 1\}$ starting at time $t=0$ in $+1$ is, at time $t=1$, in the state -1.

It is also clear from the results of Sections 4 and 5 that

$$t_L \approx \exp(\beta\tau_\beta L)$$

for large L.

Let us state our result more precisely. We denote by M the two-state space $\{-m^*(\beta), m^*(\beta)\}$ and by Y_t a continuous-time Markov chain on M with unitary jump rate for both states. Clearly the invariant measure ν of the chain Y_t is uniform over M. Let also, for any given $\varepsilon \in (0, m^*(\beta)/2)$, $t_L \equiv t_L(\varepsilon)$ be the such that

$$P^\mu(\rho(\tilde{\sigma}_0) \geqslant m^*(\beta) - \varepsilon; \rho(\tilde{\sigma}_1) \leqslant -m^*(\beta) + \varepsilon) = \frac{p}{2} \tag{6.3}$$

where p is given by (6.2) and P^μ denotes the probability over the HB-dynamics started from the equilibrium distribution $\mu^{\varnothing}_{V_L}$. Then we have:

Theorem 6.1. For any β large enough, any ε as above, and for any choice of times $t_1 < t_2 < \cdots < t_k$ and numbers $m_i \in M$, $i = 1, ..., k$,

$$\begin{aligned} \lim_{L\to\infty} & P^\mu(|\rho(\tilde{\sigma}_{t_1}) - m_1| < \varepsilon, ..., |\rho(\tilde{\sigma}_{t_k}) - m_k| < \varepsilon) \\ & = P^\nu(Y_{t_1} = m_1, ..., Y(t_k) = m_k) \end{aligned} \tag{6.4}$$

where P^ν denotes the probability of the chain Y_t with initial distribution the invariant measure ν. Moreover,

$$\lim_{L\to\infty} \frac{1}{\beta L} \log(t_L) = \tau_\beta$$

Proof. The second part follows immediately from the results of Sections 4 and 5.

As far as the first part is concerned, it is well known that

$$\lim_{L\to\infty} \mu^{\varnothing}_{V_L}(|\rho(\sigma) - m^*(\beta)| \leqslant \delta) = \tfrac{1}{2} \quad \forall\delta > 0 \tag{6.5}$$

and similarly, by the symmetry under global spin flip, for $m^*(\beta)$ replaced by $-m*(\beta)$. Hence, if the limit in the l.h.s. of (6.4) exists for fixed $t_1 < t_2 < \cdots < t_k$ and arbitrary choice of $m_i \in M$, $i = 1, ..., k$, it must be a probability measure on M^k.

We will prove (6.4) by showing that the limit along any convergent subsequence is equal to the r.h.s. of (6.4). The key step in our argument is to prove that, asymptotically as $L \to \infty$, the variables $\rho(\tilde{\sigma}_{t_1}),..., \rho(\tilde{\sigma}_{t_k})$ enjoy the Markov property. This is the content of the following proposition. For notational convenience we denote by $A_{t_i}(m_i)$ the event $|\rho(\tilde{\sigma}_{t_i}) - m_i| < \varepsilon$.

Lemma 6.1. Under the same hypotheses as Theorem 6.1, the difference

$$P^{\mu}(A_{t_1}(m_1),..., A_{t_k}(m_k))$$
$$- P^{\mu}(A_{t_1}(m_1),..., A_{t_{k-1}})) 2P^{\mu}(A_{t_{k-1}}(m_{k-1}), A_{t_k}(m_k))$$

tends to zero as $L \to \infty$.

Before giving the proof of the above key result, we complete the proof of Theorem 6.1.

Using the lemma and (6.5), it is clearly enough to prove that

$$\lim_{L \to \infty} P^{\mu}(|\rho\tilde{\sigma}_0) + m^*(\beta)| \leqslant \varepsilon; |\rho(\tilde{\sigma}_t) - m^*(\beta)| \leqslant \varepsilon)$$
$$= P^{\nu}(Y_0 = -m^*(\beta); Y_t = m^*(\beta)) = (1 - (1 - 2p)^t) \tag{6.6}$$

Let us first consider times t of the form $t = 1/m$, $m \in \mathbf{N}$, and let us define by $a(1/m)$ any limit of the l.h.s. of (6.6) computed for such t. From the lemma applied to times $t_i = i/m$, $i = 1,..., m$, and the fact that, by construction

$$a(1) = \frac{p}{2}$$

one immediately gets

$$a\left(\frac{1}{m}\right) = (1 - (1 - 2p)^{1/m}) \tag{6.7}$$

Once we know the value of $a(1/m)$ we can repeat the same argument to show that (6.6) holds also for rational times of the form $t = n/m$. In order to extend (6.6) to all times t, it is sufficient to prove, for example, that, if $\bar{a}(t)$ and $\underline{a}(t)$ denote the lim sup and lim inf of the l.h.s. of (6.6), then both of them are nondecreasing fuction of t.

For this purpose and denoting by $a_L(t)$ the l.h.s of (6.6), we immediately obtain from the lemma and (6.5) that $a_L(t+s)$ satisfies the equation

$$a_L(t+s) = a_L(t) + a_L(t)\,[1 - 4a_L(t)] + r_L \tag{6.8}$$

where $\lim_{L \to \infty} r_L = 0$.

We now observe that

$$a_L(t) \leqslant \tfrac{1}{4} + r'_L \tag{6.9}$$

where, as before, $\lim_{L \to \infty} r'_L = 0$.

In fact, again because of (6.5),

$$\begin{aligned} &P^\mu(|\rho(\tilde{\sigma}_t) - m^*(\beta)| \leqslant \varepsilon;\ |\rho(\tilde{\sigma}_0) + m^*(\beta)| \leqslant \varepsilon) \\ &\quad = \tfrac{1}{2} - P^\mu(\rho(\tilde{\sigma}_t) > 0;\ \rho(\tilde{\sigma}_0) > 0) + r'_L \end{aligned}$$

and, by the FKG property of the measure P^μ (see, e.g., ref. 13),

$$P^\mu(\rho(\tilde{\sigma}_t) > 0;\ \rho(\tilde{\sigma}_0) > 0) \geqslant P^\mu(\rho(\tilde{\sigma}_t) > 0)\ P^\mu(\rho(\tilde{\sigma}_0) > 0) = \tfrac{1}{4}$$

If we insert (6.9), we obtain

$$a_L(t+s) \geqslant a_L(t) + r_L - 4r'_L \tag{6.10}$$

Clearly (6.10) shows that $\bar{a}(t)$ and $a(t)$ are nondecreasing functions of t and thus (6.6) holds for all t.

Proof of Lemma 6.1. Using the reversibility, we can write

$$\begin{aligned} &P^\mu(A_{t_1}(m_1),..., A_{t_k}(m_k)) \\ &\quad = \int_{A_0(m_{k-1})} d\mu^{\varnothing}_{V_L}(\sigma)\ P_\sigma(A_{|t_k - t_{k-1}|}(m_k)) \\ &\qquad \times P_\sigma(A_{|t_{k-1} - t_1|}(m_1),..., A_{|t_{k-2} - t_{k-1}|}(m_{k-2})) \end{aligned} \tag{6.11}$$

where P_σ denotes the probability measure on the HB-dynamics starting from σ.

We now compare the r.h.s. of (6.11) with the quantity

$$\begin{aligned} &(\mu^{\varnothing}_{V_L}(A_0(m_{k-1})))^{-1}\ P^\mu(A_{t_1}(m_1),..., A_{t_{k-1}(m}{}^{k-1})) \\ &\quad \times P^\mu(A_{t_{k-1}}(m_{k-1}), A_{t_k}(m_k)) \end{aligned} \tag{6.12}$$

Using the stationarity of the measure P^μ and reversibility, we can write their difference as

$$[\mu^{\varnothing}_{V_L}(A_0(m_{k-1}))]^{-1} \iint_{\substack{\mathscr{A}_0(m_{k-1}) \\ \mathscr{A}_0(m_{k-1})}} d\mu^{\varnothing}_{V_L}(\sigma)\ d\mu^{\varnothing}_{V_L}(\eta) G(\eta, \sigma) \tag{6.13}$$

where

$$\begin{aligned} G(\eta, \sigma) = &\ [P_\eta(A_{|t_k - t_{k-1}|}(m_k)) - P_\sigma(A_{|t_k - t_{k-1}|}(m_k))] \\ &\times P_\sigma(A_{|t_{k-1} - t_1|}(m_1),..., A_{|t_{k-2} - t_{k-1}|}(m_{k-2})) \end{aligned} \tag{6.14}$$

Using the coupling described in Section 1 and the symmetry under global spin flip, the absolute value of (6.14) can be estimated from above by

$$2 \iint_{\substack{m(\sigma)>0\\ m(\eta)>0}} d\mu^{\varnothing}_{V_L}(\sigma)\, d\mu^{\varnothing}_{V_L}(\eta)\, P(\tilde{\sigma}_{t_k - t_{k-1}} \neq \tilde{\eta}_{t_k - t_{k-1}}) \tag{6.15}$$

which tends to zero as $L \to \infty$ because of Proposition 5.2. We have in fact that

$$t_L[t_k - t_{k-1}] \gg \exp(C\beta L^{1/2+\varepsilon})$$

because of the second part of Theorem 6.1.

The statement of the lemma now follows from (6.5) since, in (6.12), we can safely replace the factor $[\mu^{\varnothing}_{V_L}(A_0(m_{k-1})]^{-1}$ with 2. The proof is complete.

APPENDIX

In this appendix we prove Propositions 4.1–4.3. Since the proof of Proposition 3.1 is very similar, although much simpler, than that of Proposition 4.1, we decided for brevity to omit it.

Proof of Proposition 4.1. Let us fix $\varepsilon \in (0, 1/2)$ and a rectangle R,

$$R = \{x \in \mathbf{Z}^2;\ 0 \leqslant x_1 \leqslant L_1;\ 0 \leqslant x_2 \leqslant L_2\}$$

with $L_1 \geqslant L_2 \geqslant L_1^{1/2+\varepsilon}$.

If $\mathscr{A}_R^{+,+,-,+}$ is the event described in (4.12)

$$\mathscr{A}_R^{+,+,-,+} = \left\{\sigma;\ \Gamma_{\text{open}}(\sigma) \subset \left\{x \in R;\ x_2 \geqslant \frac{13L_2}{16}\right\}\right\}$$

we can write

$$\begin{aligned}
&\mu_R^{+,\delta+,-,\delta+}(\sigma(x)=1)\\
&\quad = \mu_R^{+,\delta+,-,\delta+}(\sigma(x)=1 \mid \mathscr{A}_R^{+,+,-,+})\, \mu_R^{+,\delta+,-,\delta+}(\mathscr{A}_R)\\
&\qquad + \mu_R^{+,\delta+,-,\delta+}(\sigma(x)=1 \cap (\mathscr{A}_R^{+,+,-,+})^c)
\end{aligned}$$

where $(\mathscr{A}_R^{+,+,-,+})^c$ is just the complement event.

Since

$$\mu_R^{+,\delta+,-,\delta+}(\sigma(x)=1 \mid \mathscr{A}_R) \geqslant \mu_R^{+,\delta+,+,\delta+}(\sigma(x)=1) \tag{A.1}$$

[see (4.27)], we obtain that the difference

$$\mu_R^{+,\delta+,+,\delta+}(\sigma(x)=1)-\mu_R^{+,\delta+,-,\delta+}(\sigma(x)=1)$$

can be bounded from above by

$$\mu_R^{+,\delta+,-,\delta+}((\mathscr{A}_R^{+,+,-,+})^c) \tag{A.2}$$

In order to estimate the above probability, we first observe that the event $(\mathscr{A}_R^{+,+,-,+})^c$ is a decreasing event (in the sense that its characteristic function is a nonincreasing function of the configuration). Therefore, if $\hat{R}$ is the new rectangle

$$\hat{R}=\left\{x\in\mathbf{Z}^2;\ 0\leqslant x_1\leqslant L_1,\ -\frac{L_2}{16}\leqslant x_2\leqslant L_2\right\}$$

τ is the configuration

$$\tau(x)=+1 \qquad \forall x\in\partial_{\text{ext}}\hat{R} \quad \text{with} \quad x_2\leqslant L_2-\frac{L_2}{16}$$

$$\tau(x)=-1 \qquad \text{otherwise}$$

and

$$U^{\partial\hat{R}}(x,y)=\delta \qquad \forall (x,y)\in\partial\hat{R} \quad \text{with} \quad x_1=0 \text{ or } L_1$$
$$\text{and} \quad 0\leqslant x_2\leqslant L_2-L_2/16$$

$$U^{\partial\hat{R}}(x,y)=1 \qquad \text{otherwise}$$

then

$$\mu_R^{+,\delta+,-,\delta+}((\mathscr{A}_R^{+,+,-,+})^c)\leqslant\mu_{\hat{R}}^{U^{\partial R},\tau}((\mathscr{A}_R^{+,+,-,+})^c) \tag{A.3}$$

If we denote by $\Gamma^\tau_{\hat{R},\text{open}}(\sigma)$ the (unique) open contour of $\sigma\in\Omega_{\hat{R}}$ under the τ boundary conditions described above, it is immediate to check that

$$(\mathscr{A}_R^{+,+,-,+})^c\subset(\mathscr{A}^\tau_{\hat{R}})^c\equiv\{\sigma;\ \Gamma^\tau_{\hat{R},\text{open}}(\sigma)\cap\{x\in\hat{R};\ x_2<\tfrac{13}{16}L_2\}\neq\varnothing\}$$

so that

$$\mu_{\hat{R}}^{U^{\partial R},\tau}((\mathscr{A}_R^{+,+,-,+})^c)\leqslant\mu_{\hat{R}}^{U^{\partial R},\tau}((\mathscr{A}^\tau_{\hat{R}})^c)) \tag{A.4}$$

For simplicity in the sequel we will denote the measure $\mu_R^{U^{\partial R},\tau}$ by P.

Let us now order the bonds in $\Gamma^\tau_{\hat{R},\text{open}}(\sigma)$ from left to right and let us denote by e_{k_1}, e_{k_1+n}, the smallest, respectively the largest, bond in $\Gamma^\tau_{\hat{R},\text{open}}(\sigma)$ such that no site in the portion of the exterior boundary of the left, respectively right, lateral side of $\hat{R}$ where the boundary coupling is δ is separated by one of the bonds $e\in\Gamma^\tau_{\hat{R},\text{open}}(\sigma)$ with $e\geqslant e_{k_1}$, respectively $e\leqslant e_{k_1+n}$.

We will denote by $\gamma = e_{k_1} \cdots e_{k_1+n}$ the portion of the open contour $\Gamma^{\tau}_{\hat{R},\text{open}}(\sigma)$, $\sigma \in \Omega_{\hat{R}}$, between e_{k_1} and e_{k_1+n} and by $\mathscr{F}$ the set of them.

Notice that, by construction, γ is itself an open polygonal line and that the first, respectively the last, bond in γ separates at least one site in the internal boundary of the left, respectively right, vertical side of $\hat{R}$. Moreover, if we denote by h_γ the horizontal line in $\mathbf{R}^2$ containing the middle point of the first bond e_1 of γ, then h_γ is at distance at least $L_2/16 - 1$ from the horizontal portion of the boundary of $\hat{R}$. Let also

$$d(\gamma) = \text{dist}(\gamma, h_\gamma)$$

We now define the event $\mathscr{C}$ as

$$\mathscr{C} = \left\{ \sigma ; d(\gamma) \leqslant \frac{L_2}{32} \right\} \tag{A.5}$$

Then we estimate (A.2) by

$$P((\mathscr{A}^{\tau}_{\hat{R}})^c) \leqslant P(\mathscr{C}^c) + P((\mathscr{A}^{+,+,-,+}_{\hat{R}})^c \cap \mathscr{C}) \tag{A.6}$$

Lemma A.1. Given $m > 0$, there exists $\beta(\varepsilon, m)$ such that for all $\beta \geqslant \beta(\varepsilon, m)$

$$P(\mathscr{C}^c) \leqslant \exp(-mL_1^{2\varepsilon})$$

Proof. Given γ, the set $\hat{R}$ can be written as the disjoint union of three sets:

$$\hat{R} = \Delta\gamma \cup R^+_\gamma \cup R^-_\gamma$$

where $\Delta\gamma$ has been defined in Section 1 and R^+_γ, R^-_γ lie, in a natural way, below and above γ, respectively.

Associated to the set R^+_γ we consider the partition function $Z(R^+_\gamma, U^{\partial\hat{R}}, \tau)$ with τ boundary condition and boundary coupling $U^{\partial\hat{R}}$ on $\partial_{\text{ext}} R^+_\gamma \cap \partial_{\text{ext}} \hat{R}$ and plus boundary condition on $\partial_{\text{ext}} R^+_\gamma \cap \Delta\gamma$; similarly for $Z(R^-_\gamma, U^{\partial\hat{R}}, \tau)$.

We can now write

$$P(\mathscr{C}^c) = \frac{Z(\hat{R}, U^{\partial\hat{R}}, +)}{Z(\hat{R}, U^{\partial\hat{R}}, \tau)} \sum_{\substack{\gamma;\, d(\gamma) \geqslant L_2/32 \\ \gamma \in \mathscr{F}}} \exp(-2\beta|\gamma|)$$

$$\times \frac{Z(R^+_\gamma, U^{\partial\hat{R}}, \tau)\, Z(R^-_\gamma, U^{\partial\hat{R}}, \tau)}{Z(\hat{R}, U^{\partial\hat{R}}, +)} \tag{A.7}$$

Unfortunately, we cannot yet use the cluster expansion described in Section 1 to simplify the above ratio of partition functions, since, although in $Z(R_\gamma^+, U^{\partial\hat{R}}, \tau)$ the boundary condition is constantly equal to $+1$ because of (a) in the definition of γ, so that

$$Z(R_\gamma^+, U^{\partial\hat{R}}, \tau) = Z(R_\gamma^+, U^{\partial\hat{R}}, +)$$

in $Z(R_\gamma^-, U^{\partial\hat{R}}, \tau)$ the lateral boundary condition may change sign. However, a trivial and rough comparison shows that

$$\exp(-4\beta\,\delta L_2) \leqslant \frac{Z(R_\gamma^-, U^{\partial\hat{R}}, \tau)}{Z(R_\gamma^-, U^{\partial\hat{R}}, -)} \leqslant \exp(+4\beta\,\delta L_2) \tag{A.8}$$

Therefore the r.h.s. of (A.7) is bounded from above by

$$\exp(+8\beta\,\delta L_2) \Bigg[\sum_{\substack{\gamma;\, d(\gamma) \geqslant L_2/32 \\ \gamma \in \mathscr{F}}} \exp(-2\beta|\gamma|) \frac{Z(R_\gamma^+, U^{\partial\hat{R}}, +) Z(R_\gamma^-, U^{\partial\hat{R}}, -)}{Z(\hat{R}, U^{\partial\hat{R}}, +)} \Bigg]$$
$$\times \Bigg[\sum_{\substack{\gamma \subset \hat{R} \\ \gamma \in \mathscr{F}}} \frac{Z(R_\gamma^+, U^{\partial\hat{R}}, +) Z(R_\gamma^-, U^{\partial\hat{R}}, -)}{Z(\hat{R}, U^{\partial\hat{R}}, +)} \Bigg]^{-1} \tag{A.9}$$

We observe at this point that each one of the partition functions

$$Z(\hat{R}, U^{\partial\hat{R}}, +), \qquad Z(R_\gamma^+, U^{\partial\hat{R}}, +), \qquad Z(R_\gamma^-, U^{\partial\hat{R}}, -)$$

can be written as in (1.10), with weights that satisfy the condition of Proposition 1.1 with constant $\alpha = 1/2$. Therefore, following ref. 4, we can apply Proposition 1.1 to write

$$\frac{Z(R_\gamma^+, U^{\partial\hat{R}}, +) Z(R_\gamma^-, U^{\partial\hat{R}}, -)}{Z(R, U^{\partial\hat{R}}, +)} = \exp\Bigg[- \sum_{\substack{\Lambda \subset \hat{R} \\ \Lambda \cap \Delta\gamma \neq \varnothing}} \Phi^{U^{\partial\hat{R}}, +}(\Lambda) \Bigg] \tag{A.10}$$

so that (A.9) becomes

$$\exp(16\beta\,\delta L_2) \Bigg\{ \sum_{\substack{\gamma;\, d(\gamma) \geqslant L_2/32 \\ \gamma \in \mathscr{F}}} \exp(-2\beta|\gamma|) \exp\Bigg[- \sum_{\substack{\Lambda \subset \hat{R} \\ \Lambda \cap \Delta\gamma \neq \varnothing}} \Phi^{U^{\partial\hat{R}}, +}(\Lambda) \Bigg] \Bigg\}$$
$$\times \Bigg\{ \sum_{\substack{\gamma \subset \hat{R} \\ \gamma \in \mathscr{F}}} \exp(-2\beta|\gamma|) \exp\Bigg[- \sum_{\substack{\Lambda \subset \hat{R} \\ \Lambda \cap \Delta\gamma \neq \varnothing}} \Phi^{U^{\partial\hat{R}}, +}(\Lambda) \Bigg] \Bigg\}^{-1} \tag{A.11}$$

We can use at this point two basic results in ref. 4 (see the proposition and the theorem in Sections 4.14 and 4.16, respectively) to conclude that, since

$L_1 \geqslant L_2 \geqslant L_1^{1/2+\varepsilon}$, for any given $m>0$, the ratio between the two sums in (A.11) is smaller than

$$\exp\left(-m\frac{L_2^2}{L_1}\right) = \exp(-mL_1^{2\varepsilon})$$

provided that β is large enough.
The lemma is proved.

We now turn to the estimate of the second term in the r.h.s of (A.6), $P((\mathscr{A}_{\hat{R}}^{\tau})^c \cap \mathscr{C})$.

Let $l = L_2/16$ (we are assuming for simplicity that $L_2/16$ is an integer) and let, for $i = 0,..., N = 32$, R_i be the rectangle

$$R_i = \left\{x \in R;\ 0 \leqslant x_1 \leqslant L_1, -\frac{L_2}{16} + i\frac{l}{2} \leqslant x_2 \leqslant -\frac{L_2}{16} + (i+2)\frac{l}{2}\right\}$$

Then we define P_i as

$$P_i = P(\{\gamma \subset R_i\} \cap \mathscr{C})$$

and we estimate from above $P((\mathscr{A}_{\hat{R}}^{\tau})^c \cap \mathscr{C})$ by

$$P((\mathscr{A}_{\hat{R}}^{\tau})^c \cap \mathscr{C}) \leqslant \sum_{i=1 \cdots N-5} P(\{\gamma \subset R_i\} \cap \mathscr{C}) \tag{A.12}$$

In (A.12) we used the fact that, if the event $((\mathscr{A}_{\hat{R}}^{\tau})^c \cap \mathscr{C}$ occurs, then, by construction, γ is entirely contained in some R_i because of $\mathscr{C}$, with the index $i \neq N,..., N-4$ because of $(\mathscr{A}_{\hat{R}}^{\tau})^c$ and $i \neq 0$ again because of $\mathscr{C}$.

In order to estimate each term in the r.h.s. of (A.12), we proceed in a slightly different way depending on whether $i = 1$ or $i > 1$, the reason being that in the case $i = 1$ the polygonal line γ is very close to the discontinuity point of the lateral boundary coupling.

Let us first consider the case $i \geqslant 2$. In this case we bound from above the ratio P_i/P_{N-2} uniformly in $i = 2,..., N-5$. If we use the representation $(A.7)$ for the probability of a given γ, we may write

$$\frac{P_i}{P_{N-2}} = \frac{\sum_{\gamma \subset R_i;\, d(\gamma) \leqslant L_2/32} \exp(-2\beta|\gamma|)\, Z(R_\gamma^+, U^{\partial\hat{R}}, \tau)\, Z(R_\gamma^-, U^{\partial\hat{R}}, \tau)}{\sum_{\gamma \subset R_{N-2};\, d(\gamma) \leqslant L_2/32} \exp(-2\beta|\gamma|)\, Z(R_\gamma^+, U^{\partial\hat{R}}, \tau)\, Z(R_\gamma^-, U^{\partial\hat{R}}, \tau)} \tag{A.13}$$

Given $\gamma \subset R_i$, let $F_i(\gamma)$ be its image under a vertical translation in $\mathbf{R}^2$ by an amount $(N-2-i)\, l/2$.

Then clearly F_i establish a bijection between the γ in R_i and those in R_{N-2}, so that the r.h.s. is estimated from above by

$$\sup_{\gamma \subset R_i;\, d(\gamma) \leqslant L_2/32} \frac{Z(R_\gamma^+, U^{\partial \hat{R}}, \tau)\, Z(R_\gamma^-, U^{\partial \hat{R}}, \tau)}{Z(R_{F_i(\gamma)}^+, U^{\partial \hat{R}}, \tau)\, Z(R_{F_i(\gamma)}^-, U^{\partial \hat{R}}, \tau)} \tag{A.14}$$

which we write as

$$\sup_{\gamma \subset R_i;\, d(\gamma) \leqslant L_2/32} \frac{Z(R_\gamma^+, U^{\partial \hat{R}}, +)\, Z(R_\gamma^-, U^{\partial \hat{R}}, -)}{Z(R_{F_i(\gamma)}^+, U^{\partial \hat{R}}, +)\, Z(R_{F_i(\gamma)}^-, U^{\partial \hat{R}}, -)} \times \frac{Z(R_\gamma^-, U^{\partial \hat{R}}, \tau)\, Z(R_{F_i(\gamma)}^-, U^{\partial \hat{R}}, -)}{Z(R_\gamma^-, U^{\partial \hat{R}}, -)\, Z(R_{F_i(\gamma)}^-, U^{\partial \hat{R}}, \tau)} \tag{A.15}$$

Let us consider the first ratio

$$\frac{Z(R_\gamma^+, U^{\partial \hat{R}}, +\tau)\, Z(R_\gamma^-, U^{\partial \hat{R}}, -)}{Z(R_{F_i(\gamma)}^+, U^{\partial \hat{R}}, +)\, Z(R_{F_i(\gamma)}^-, U^{\partial \hat{R}}, +)} \tag{A.16}$$

If we divide numerator and denominator by $Z(\hat{R}, U^{\partial \hat{R}}, +)$ and we use $(A.10)$, we get that $(A.16)$ is equal to

$$\exp\left[\sum_{\substack{\Lambda \subset \hat{R} \\ \Lambda \cap \Delta\gamma \neq \varnothing}} \Phi^{U^{\partial \hat{R}}, +}(\Lambda) + \sum_{\substack{\Lambda' \subset \hat{R} \\ \Lambda' \cap \Delta F_i(\gamma) \neq \varnothing}} \Phi^{U^{\partial \hat{R}}, +}(\Lambda')\right] \tag{A.17}$$

Notice that, for any pair $\Lambda \subset \hat{R}$, $\Lambda' \subset \hat{R}$ that intersect neither the horizontal part of $\partial \hat{R}$ nor the lateral portion where $U^{\partial \hat{R}} = 1$ and are one the translate of the other

$$\Lambda' = F_i(\Lambda)$$

we have

$$\Phi^{U^{\partial \hat{R}}, +}(\Lambda') = \Phi^{U^{\partial \hat{R}}, +}(\Lambda) \tag{A.18}$$

by the very definition of the coefficients $\Phi^{U^{\partial \hat{R}}, +}(\Lambda)$.

Therefore the difference between the two sums appearing in (A.17) becomes simply

$$\sum_{\Lambda, \Lambda'}^{\gamma} \Phi^{U^{\partial \hat{R}}, +}(\Lambda') - \Phi^{U^{\partial \hat{R}}, +}(\Lambda) \tag{A.19}$$

where $\sum_{\Lambda, \Lambda'}^{\gamma}$ is a shorthand notation for the sum over all pairs, Λ, Λ' which intersect $\Delta\gamma$ and $\Delta F_i(\gamma)$, respectively, and are such that one of the above two requirements is violated by Λ or $F_i(\Lambda)$ and by Λ' or $F_i^{-1}(\Lambda')$, where F_i^{-1} is the inverse of F_i.

Since $\gamma \subset R_i$, $F_i(\gamma) \subset R_{N-2}$, and $d(\gamma) \leqslant L_2/32$, we can bound from above (A.19), uniformly in $i = 2,..., N-5$, by

$$\sum_{\substack{\Lambda \cap R_i \neq \varnothing \\ \Lambda \cap \{\partial \hat{R} \setminus \partial R_i\} \neq \varnothing}} |\Phi^{U^{\partial \hat{R}},+}(\Lambda)| + \sum_{\substack{\Lambda' \cap R_{N-2} \neq \varnothing \\ \Lambda' \cap \{\partial \hat{R} \setminus \partial R_{N-2}\} \neq \varnothing}} |\Phi^{U^{\partial \hat{R}},+}(\Lambda')| \leqslant C \tag{A.20}$$

for a suitable constant C independent of $\hat{R}$.

In (A.20) we used the exponential decay of $\Phi^{U^{\partial \hat{R}},+}(\Lambda)$ in the "size" $d(\Lambda)$ of Λ, the fact that the distance between the horizontal part of the boundaries of $\hat{R}$, R_i, R_{N-2} is, by construction and because $d(\gamma) \leqslant L_2/32$, at least $L_2/32$, and the fact that the boundary coupling $U^{\partial \hat{R}}$ is equal to δ on the lateral boundary of R_i, $i = 2,..., N-5$, by construction.

Let us consider the second ratio in (A.15),

$$\frac{Z(R_\gamma^-, U^{\partial \hat{R}}, \tau)\, Z(R_{F_i(\gamma)}^-, U^{\partial \hat{R}}, -)}{Z(R_\gamma^-, U^{\partial \hat{R}}, -)\, Z(R_{F_i(\gamma)}^-, U^{\partial \hat{R}}, \tau)}$$

Using the Jensen inequality, we obtain

$$\begin{aligned} \frac{Z(R_\gamma^-, U^{\partial \hat{R}}, \tau)}{Z(R_\gamma^-, U^{\partial \hat{R}}, -)} &\leqslant \exp\left(2\beta\delta \sum_{(x,y) \in \partial R_\gamma^-;\, U^{\partial \hat{R}}(x,y) = \delta} \langle \sigma(x) \rangle^\tau \right) \\ \frac{Z(R_{F_i(\gamma)}^-, U^{\partial \hat{R}}, -)}{Z(R_{F_i(\gamma)}^-, U^{\partial \hat{R}}, \tau)} &\leqslant \exp\left(-2\beta\delta \sum_{(x,y) \in \partial R_{F_i(\gamma)}^-;\, U^{\partial \hat{R}}(x,y) = \delta} \langle \sigma(x) \rangle^- \right) \end{aligned} \tag{A.21}$$

where $\langle \sigma(x) \rangle^\tau$ is a shorthand notation for the average of the spin $\sigma(x)$ in the Gibbs measure $\mu_{R_\gamma^-}^{U^{\partial \hat{R}},\tau}$ and similarly for $\langle \sigma(x) \rangle^-$.

A simple Peierls argument shows that

$$\langle \sigma(x) \rangle^\tau \leqslant -1 + k \qquad \forall x \in \partial_{\text{int}} R_\gamma^-; \quad U^{\partial \hat{R}}(x, y) = \delta$$

with $k \to 0$ as $\beta \to \infty$, so that from (A.21) we obtain that

$$\frac{Z(R_\gamma^-, U^{\partial \hat{R}}, \tau)\, Z(R_{F_i(\gamma)}^-, U^{\partial \hat{R}}, -)}{Z(R_\gamma^-, U^{\partial \hat{R}}, -)\, Z(R_{F_i(\gamma)}^-, U^{\partial \hat{R}}, \tau)} \leqslant \exp(-\beta\,\delta l + 2\beta\,\delta k L_2) \tag{A.22}$$

Finally, combining (A.20) and (A.22), we obtain that

$$P_i \leqslant C \exp(-\beta\,\delta l + 2\beta\,\delta k L_2) \quad \forall i = 2,..., N-5 \tag{A.23}$$

Let us now treat the case $i=1$ by estimating from above the ratio P_1/P_{N-1}. We define the map F_1 to be simply the clockwise rotation of π around the center of the rectangle $\hat{R}$ and we proceed as before. In this case, by symmetry, the ratio (A.16) with $F_i(\gamma)$ replaced by $F_1(\gamma)$ is equal to one and the rest of the argument does not change.

In conclusion, since l is proportional to L_2 and k is very small for large β, we get that, for any $m>0$,

$$P(\mathscr{A}^c_{\hat{R}} \cap \mathscr{C}) \leqslant \exp(-mL_1^{\varepsilon}) \tag{A.24}$$

Thus, combining together (A.3), (A.4), Lemma A.1, and (A.24), we get the first part of the proposition.

The second part follows immediately by a standard Peierls argument. The proposition is proved.

Proof of Proposition 4.2. Following the proof of Proposition 4.1, let τ be the configuration

$$\tau(x) = +1 \qquad \forall x \in \partial_{\text{ext}} R \quad \text{with} \quad x_2 \leqslant L_2 - \frac{L_2}{16}$$

$$\tau(x) = -1 \qquad \text{otherwise}$$

and let

$$U^{\partial R}(x, y) = \delta \qquad \forall (x, y) \in \partial R \quad \text{with} \quad 0 \leqslant x_2 < L_2$$

$$U^{\partial R}(x, y) = 1 \qquad \text{otherwise}$$

Let also S be the cigar-shaped neighborhood of the segment of the horizontal line at height $L_2 - L_2/16$ and joining the two vertical sides of R:

$$S = \left\{ (x_1, x_2) \in R;\ \left| x_2 - \left(L_2 - \frac{L_2}{16} \right) \right| \leqslant \left(\frac{x_1(L_1 - x_1)}{L_1} \right)^{(1+\varepsilon)/2} \right\} \tag{A.25}$$

Notice that, for large values of L_1, the region S is at distance at least $L_2/32$ from the upper horizontal side of R.

Then it is immediate to see that

$$\mu_R^{\delta+,\delta+,-,\delta+}((\mathscr{A}_R^{+,+,-,+})) \geqslant \mu_R^{U^{\partial R},\tau}(\mathscr{S}^{\tau}_R) \tag{A.26}$$

where

$$\mathscr{S}^{\tau}_R = \{\sigma;\ \Gamma^{\tau}_{R,\text{open}} \subset S\}$$

Let $\mathscr{F}_R$ be the set of all possible configurations of $\Gamma^{\tau}_{R,\text{open}}$. As in (A.7), we write

$$\mu_R^{U^{\partial R},\tau}(\mathscr{S}^{\tau}_R)=\frac{Z(R,U^{\partial R},-)}{Z(R,U^{\partial R},\tau)} \times \sum_{\Gamma\in\mathscr{F}_R;\,\Gamma\subset S}\exp(-2\beta|\Gamma|)\frac{Z(R_\Gamma^+,U^{\partial R},\tau)\,Z(R_\Gamma^-,U^{\partial R},\tau)}{Z(R,U^{\partial R},-)} \tag{A.27}$$

where R_Γ^+ and R_Γ^- are defined as in the proof of Proposition 4.1.

The ratio $Z(R,U^{\partial R},-)/Z(R,U^{\partial R},\tau)$ is clearly bounded from below by

$$\exp[-\delta(2L_2+L_1)\beta] \tag{A.28}$$

Notice that, since the polygonal line Γ is contained in the region S, the boundary conditions in the partition functions $Z(R_\Gamma^+,U^{\partial R},\tau)$ and $Z(R_\Gamma^-,U^{\partial R},\tau)$ are, by construction, $+$ and $-$, respectively. Therefore, using the representation (1.10), we can write the ratio in the second factor in (A.27) as

$$\frac{Z(R_\Gamma^+,U^{\partial R},\tau)\,Z(R_\Gamma^-,U^{\partial R},\tau)}{Z(R,U^{\partial R},-)}=\exp\Bigg[-\sum_{\substack{\Lambda\subset R\\ \Lambda\cap\Delta\gamma\neq\varnothing}}\Phi^{U^{\partial R},+}(\Lambda)\Bigg] \tag{A.29}$$

Using Proposition 1.1 and the fact that the polygonal line Γ is contained in the region S, it is easy to see that

$$\sum_{\substack{\Lambda\subset R\\ \Lambda\cap\Delta\gamma\neq\varnothing}}\Phi^{U^{\partial R},+}(\Lambda)\leqslant\sum_{\Lambda\cap\Delta\gamma\neq\varnothing}\Phi^{+}(\Lambda)+C_0$$

for a suitable constant C_0 independent of L_1.

Thus (A.27) can be estimated from below by

$$\exp(-C_0-3\delta\,\beta L_1)\sum_{\Gamma\in\mathscr{F}_R;\,\Gamma\subset S}\exp\Bigg[-2\beta|\Gamma|-\sum_{\substack{\Lambda\in R\\ \Lambda\subset\Delta\gamma\neq\varnothing}}\Phi^{+}(\Lambda)\Bigg] \tag{A.30}$$

We use at this point the fundamental result of ref. 4 (see Section 4.16), which says that

$$\sum_{\Gamma\in\mathscr{F}_R;\,\Gamma\subset S}\exp\Bigg[-2\beta|\Gamma|-\sum_{\substack{\Lambda\subset R\\ \Lambda\cap\Delta\gamma\neq\varnothing}}\Phi^{U^{\partial R},+}(\Lambda)\Bigg] \geqslant\exp\{-\beta L_1\tau_\beta-C[\log(L_1)]^{\max(6,2/\varepsilon)}\} \tag{A.31}$$

If we combine together (A.26)–(A.28), (A.30), and (A.31), we finally get the result.

Proof of Proposition 4.3. It is easy to show that the expression appearing in part (a) is bounded from above by

$$|F_2|_\infty \sum_{x \in \partial_{\text{ext}}(\text{bottom side of } Q_2)} \mu_{Q_1}^{\delta+,\delta+,-,\delta+}(\tau(x)=1 \mid \mathscr{A}_{Q_1}^{+,+,-,+}) - \mu_{R_{n+1}\cup Q_{n+1}}^{\delta+,\delta+,+,\delta+}(\tau(x)=1) \tag{A.32}$$

By monotonicity

$$\mu_{Q_1}^{\delta+,\delta+,-,\delta+}(\tau(x)=1 \mid \mathscr{A}_{Q_1}^{+,+,-,+}) \leqslant \mu_{R_1}^{\delta+,\delta+,+,\delta+}(\tau(x)=1)$$

A standard Peierls argument shows that, for each $x \in \partial_{\text{ext}}(\text{bottom side of } Q_2)$ and any given positive m

$$0 \leqslant \mu_{R_1}^{\delta+,\delta+,+,\delta+}(\tau(x)=1) - \mu_{R_{n+1}\cup Q_{n+1}}^{\delta+,\delta+,+,\delta+}(\tau(x)=1) \leqslant \exp(-mL^{1/2+\varepsilon}) \tag{A.33}$$

provided that β is large enough.

Clearly (A.33) proves part (a), since

$$|F_2|_\infty \leqslant 2L^2$$

Part (b) follows immediately from part (a) and Proposition 4.1 applied to the rectangle $R_{n+1} \cup Q_{n+1}$. The proposition is proved.

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